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Study the Artin-Wedderburn Theorem

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ABSTRACT

A simple left artinian ring is isomorphic to the ring of matrices over a division ring, according to the well-known Wedderburn-Artin theorem. We provide a brief, self-contained proof without the need for modules. In noncommutative ring theory, the Wedderburn-Artin theorem is crucial. A brief, self-contained proof that only takes basic knowledge of rings is provided. We will talk about it in this paper. The Artin-Wedderburn theorem should be studied.

KEYWORDS: *Artin-Wedderburn Theorem, Ring, Proof, Modules, Noncommutative, Elementary Facts, Algebras, Descending Chain, Division Ring*

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INTRODUCTION

The classification theorem for semisimple rings and semisimple algebras is known as the Wedderburn–Artin theorem in algebra.

According to the theorem, for certain integers n_i, a (Artinian) semisimple ring R is isomorphic to a product of finitely many n_i-by-n_i matrix rings over division rings D_i. As long as the index i is permuted, both are uniquely determined. Specifically, an n-by-n matrix ring over a division ring D is isomorphic to any simple left or right Artinian ring, where n and D are uniquely determined. [1]

The structure of associative Artinian rings without nilpotent ideals is completely described by this theorem. If and only if RR is the direct sum of a finite number of ideals, each of which is isomorphic to a complete matrix ring of finite order over a suitable skew-field, then an associative ring RR has the minimum condition for right ideals and has no nilpotent ideals; this decomposition into a direct sum is distinct from the ordering of its terms. J. Wedderburn initially derived this theorem for finite-dimensional algebras over a field, and E. Artin proved it in its ultimate form. [2]

Artinian rings, also known as Artin rings in abstract algebra, are rings that satisfy the descending chain condition on (one-sided) ideals, which states that there isn't an endless descending sequence of ideals. Emil Artin, who initially found that the descending chain condition for ideals simultaneously generalizes finite rings and rings that are finite-dimensional vector spaces over fields, is credited with naming Artinian rings. An comparable concept, the minimum condition, can be used to repeat the definition of Artinian rings in place of the descending chain requirement. [3]

Specifically, a ring is Artinian or two-sided Artinian if it is both left and right Artinian, Artinian if it is right Artinian, and Artinian if it is left Artinian if it meets the descending chain condition on left ideals. Although the left and right definitions of commutative rings coincide, they are generally different. [4]

Every simple Artinian ring is described by the Wedderburn–Artin theorem as a ring of matrices over a division ring. This suggests that if and only if a simple ring is right Artinian, it is left Artinian. [5]

Theorem:
Let R be an Artinian semisimple ring. The Wedderburn-Artin theorem says that R is isomorphic to a product of finitely numerous n_i-by-n_i matrix rings $M_{ni}(D_i)$ over division rings D_i, for some integers n_i, both of which are uniquely determined up to permulation of the index i.

A variant of the Wedderburn–Artin theorem for algebras over a field k exists as well. Each D_i in the aforementioned assertion is a finite-dimensional division algebra over k if R is a finite-dimensional semisimple k-algebra. A finite extension of k could serve as the center of each D_i instead of k.

It should be noted that D need not be contained in E if R is a finite-dimensional simple algebra over a division ring E. For instance, finite-dimensional simple algebras over the real numbers are matrix rings over the complex numbers. [6]

Proof:
The Wedderburn–Artin theorem has several proofs. One popular contemporary one adopts the following strategy. [7]

Assume the ring R is semisimple. The right R-module RR is isomorphic to a finite direct sum of simple modules (which are the same as R's minimum right ideals). Write the direct sum as

$$RR \cong \oplus_{i=1}^{m} I_i^{\oplus ni}$$

The Ii are mutually nonisomorphic simple right R-modules, with the ith one having multiplicity n_i. This results in an isomorphism between endomorphism rings.

$$End\ (RR \cong \oplus_{i=1}^{m} End\,(I_i^{\oplus ni})$$

And we can identify $End(I_i^{\oplus ni})$ with a ring of matrices

$$End(I_i^{\oplus ni}) \cong M_{ni}(End(I_i))$$

Because Ii is simple, the endomorphism ring *End(Ii)* is a division ring according to Schur's lemma. Since $R \cong End(R_R)$ *we include*

$$R \cong \oplus_{i=1}^{m} M_{ni}\ (End(I_i)).$$

We used right molecules because $R \cong End(R_R)$. If we used left modules, R would be isomorphic to the opposite algebra of $End(R_R)$, but the proof would still be valid. Decomposition of a module provides a greater framework for this proof. For the evidence of an important specific instance, see Simple Artinian rings.

Review of Literature:
Joseph Wedderburn provided the first proof in 1905 and later proved the theorem in two more ways. Leonard Eugene Dickson accepted Wedderburn's precedence when he presented another proof soon after Wedderburn's initial proof. Nevertheless, as mentioned in (Parshall 1983), Wedderburn's initial proof was flawed due to a gap, and his following proofs were only presented after he had read Dickson's accurate proof. Parshall contends that Dickson should be given credit for the first accurate proof based on this. [8]

Objectives:
- To Study the Artin-Wedderburn Theorem
- To Study on Weddernburn -Artin Theorem for Rings

Research Methodology:
This study's overall design was exploratory. The research paper is an endeavor that relies on secondary data collected from reliable sources, including newspapers, textbooks, journals, and the internet. The research design of the study is mostly descriptive.

Result and Discussion:
A STUDY ON WEDDERNBURN -ARTIN THEOREM FOR RINGS:
Ref. [9] states that the topic of figuring out the structure of rings and algebras with all of their modules being the direct sum of certain cyclic modules has a lengthy history. showed that every module over a finite dimensional K – algebra A is a direct sum of simple modules if and only if

$$A \cong \prod_{i=1}^{m} M_{n_i}(D_i) \text{ where } m, n_1, \ldots\ldots, \in N \text{ and each } D_i \text{ is}$$

Finite dimensional division algebra over k.

One of the two sections of the first Wedderburn Artin theorem, which deals with finite simple algebra, has been noted from the study of [10]. We let R represent an identity ring. If the ring R is a direct sum of minimal left ideals, then the ring is referred to as a left semisimple ring. Here

$$R = \oplus_{i\in s} I_i$$

This paper's primary outcome is listed below.

Theorem 2.1 states that if R is any left semisimple ring, then

$$R \cong M_{n1}(D_1) x \ldots x M_{nr}(D_r)$$

Where each D_i is a division ring and $M_n(D)$ denotes the ring of $n \times n$ matrices over D

Theorem 2.2: Let R be a finite-dimensional algebra with identity over the field F. If R is a semisimple ring, it is isomorphic to $M_i(D)$ for any integer $n \geq 1$ and finite dimensional algebra D over F. R determines the integer n uniquely, but D is unique up to isomorphism.

We start by proving the aforementioned theorems in order to arrive at our primary result.

Proof of theorem 2.1:

Proof of theorem 2.1: Using the work of [2], we express R as the direct sum of minimal left ideals, and then regroup the summand according to their R isomorphic type as

$$R = \oplus^r_{i \in s} n_j V_j,$$

Where $n_i V_j$ is the direct sum of n_j submodules R isomorphic to V_j and where $V_i \neq V_j$ for $i \neq j$. The isomorphison is one of unital left R modules. Now put $D_i^0 = End_{R(Vi)}$. This is a division ring by Schur's Lemma as it was proved in [5] research.

We obtain an isomorphism of rings of rings by applying proposition 10.14 of [11].

$$R^0 \simeq End_R R \simeq Hom_R(\oplus_{i=1}^r n_i V_j, \oplus_{j=1}^r n_j V_j \ldots\ldots 2.1$$

Define a map $p_i : \oplus_{i=1}^r n_i V_j \to \oplus_{j=1}^r n_j V_j$

To be the i^{th} *projection and* $p_i : n_i V_j \to \oplus_{j=1}^r n_j V_j$ to be the i^{th} inclusion. Let us see that the right side of 2.1 is isomorphic as a ring to $\prod_i End_{R(n_j V_j)}$ via mapping

$$f \to (p_1 f q_1, \ldots p_r f q_r).$$

What is to be presented here is that $p_1 f q_1 = 0 for i \neq j$. Now $p_1 f q_1$ is a member of $Hom_R(n_j V_j, n_j V_j)$.

Accordingly, using (2.1) above, we get

$$R^0 \simeq \prod_{i=1}^r Hom_R(n_i V_i, n_i V_i) = \prod_{i=1}^r End_R(n_i V_i)$$

$$\cong \coprod_{i=1}^r M_{ni}\left(End_R(V_i)\right) \text{ by corollary 10.13 of [3]}$$

$$\cong \coprod_{i-1}^r M_{ni}(D_i^0) \text{ by definition of } D_i^0$$

To reverse the order of multiplication in R^0, use a transpose map. $M_{n_1'}(D_i^0)$, we conclude that $R \cong \prod_{i=1}^r M_{ni}(D_i)$. This demonstrates the existence of the decomposition in the supplied theorem of the main result. We still have to identify the simple left R module and establish it's a suitable unique statement, as contained in [2]. Recalled, in example (i) above, we have decomposition $M_{ni}(D_i) \cong D_i^{ni} \ldots \oplus D_i^{ni}$ of left $M_{ni}(D_i)$ modules, and each term (D_i^{ni}) as simple left $M_{ni}(D_i)$ module. The decomposition proved allows the researchers to regard each term (D_i^{ni}) as simple left R module, $1 \leq i \leq r$. each of these modules is acted upon by a different coordinate of R, and hence we have projected at least r non-isomorphic simple left R module s as in [3]. The researcher added that any simple R module must be a quotient of R by a maximal left ideal, as we observed in example (ii) hence a composition factor as consequences of the Jordan- Holder theorem in [6]. There are only r non-isomorphic such $V_{j's}$,and we conclude that the number of simple left R modules, up to isomorphism, is exactly r.

For uniqueness, we consider [3] which opined that supposed that $M_{n_1'}(D_i')$ module up to isomorphison, and regard V_i' as a simple left R module. Then we have

$R \cong \oplus_{j-1}^s n_j' n_j'$ as left R modules. By the Jordan Holder Theorem we must have r=s and, after a suitable numbering, $n_i = n_i'$and $V_i \cong V_i'$ for $1 \leq i \leq r \leq$. Thus we have ring isomorphison

$R \simeq M_{n_1'(D_i)} x \ldots x M_{n_s'(D_s)}$ as rings. Let $V_i = (D_j')^{n_1'}$ be the unique simple left

$$D_i' \cong End_{M_{n_i'}(D_i')} \text{By lemma 2.1 of [3]}$$

$$\simeq End_{R(V_i')}$$

$$\simeq End_{R(V_i)} \quad \text{Since } V_i \cong V_i'$$

$$\cong D_i^0$$

Reversing the order of multiplication gives $D_i' \cong D_i$, and hence proved.

Proof of theorem 2.2: from the work of [2] and by finite dimensionality, R has a minimal left ideal V. For r in R, form the set Vr. This is a left ideal, and we claim that it is minimal or is 0. In fact, the function $v \to vr$ is R linear from V onto Vr. Since V is simple as a left R module, Vr is simple or 0. The sum $I = \Sigma_r$ *with* $r \neq 0$ Vr is a two-sided ideal in R, and it is not 0 because $V1 \neq 0$. Since R is simple, $I=R$. Then the left R module R is exhibited as the sum of simple left R modules and is therefore semisimple. The isomorphism with $M_n(D)$ and the uniqueness now follow from Theorem 2.1 above.

Conclusion:

A theorem of structure is the Artin-Wedderburn theorem. It indicates that the structure of a ring is strictly limited if it satisfies some particular criteria, and that property is that it must be a finite direct product of matrix rings over division rings. These rings are really well-understood; we can work with them in a very precise manner and say many things about them that may be difficult to say about general rings. Knowing that semi-simplicity suggests that a ring is very easy to understand is hence quite convenient. When the free left R-module underlying R is a sum of simple R-modules, the proof yields a very nice main conclusion, which is that a ring R with unity is semi simple, or left semi simple to be exact. The Wedderburn-Artin theorem is referred to as Wedderburn's theorem when R is simple, according to his brief proof of the theorem.

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