Planar Signed Graphs Whose Circular Chromatic Number Is Between 14/3 and 6

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ABSTRACT

In 2020, Naserasr, Wang and Zhu in [6] constructed a signed planar graph (G,σ) that has a $\chi_c(G) = 4 + \frac{2}{3}$. Moreover every planar signed graph (G,σ) has a $\chi_c(G) < 6$. This paper continues to reinforce this result by using a method allowing to construct, for each rational $4 + \frac{2}{3} \leq r < 6$, a signed planar graph (G,σ) whose $\chi_c(G) = r$. Considering the previous findings of Moser [2] in 1997 and Zhu [17] in 2001, this clearly indicates that any rational r is the circular chromatic number of a planar signed graph.

KEYWORDS: *Planar signed graphs, circular coloring, circular-r-coloring*

MSC Classification: 05C15, 05C10, 05C22

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1. INTRODUCTION

According to Kaiser, and Rollov´a in [9], a signed graph is a graph G together with a signature σ: $E(G) \rightarrow \{+1,-1\}$ which assigns to each edge a sign. A positive (resp. negative) edge between $u,v \in V$, denoted uv^+ (resp. uv^-).

Studied by many authors since 1980, the notion of coloring of signed graphs originated from Zaslavsky. The author begins his study by coloring the vertices of signed graphs [7, 8]. Thus, he establishes a correspondence g: $V(G) \rightarrow \{\pm l, \pm(l-1)..., \pm 1, 0\}$ such that any edge uv of G, $g(u) \not\models \sigma g(v)$. The coloring containing $(2l+1)$ labels taken in the set $\{-l,-(-l-1),...,-1,0,1,...,(l-1),l\}$

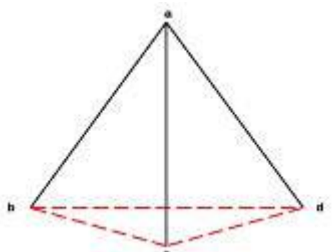

Fig. 1 Signed graph: The solid and the dashed lines denote the positive edge and negative edge, respectively

is called a l-coloring of signed graphs. The one containing $(2l)$ labels taken into the set $l,-(l-1),...,-1,0,1,...,(l-1),l\}$ is called an l-coloring without zero.

In 1988, the concept of circular chromatic number, previously called star chromatic number of a graph G, was born by Vince (see [1]). He notes it $\chi_c(G)$.

He generalized this concept naturally from the chromatic number of a graph. The concept of "circular chromatic number" appeared in [14]. The previous definition appeared in [17].

Among the most important results of the circular chromatic number of a graph G since 1988, we have the following famous relation: for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. And by extension we have: $\chi(G) = \lceil \chi_c(G) \rceil$.

According to Zhu in [17, 18], for a real number $r \geq 1$, C_r is a circle of circumference r. Consider two points u,v belonging to C_r, the distance $d_{(\bmod r)}(u,v)$ between u and v defined by Zhu in 2006 is the distance of the smallest arc of C^r connecting u and v: $d_{(\bmod r)}(u,v) = \min\{abs(u-v), r - abs(u-v)\}$. Assuming the point α belongs to C_r, the unique point belonging to C_r of distance $\frac{r}{2}$ from this point is called the antipode. It is denoted by $\bar{\alpha}$.

According to Zhu, the notion of a circular r-coloring of (G,σ) is a corresponding $g: V(G) \rightarrow C_r$ such that for each positive edge uv, $d_{(\bmod r)}(g(u),g(v)) \geq 1$, and for each negative edge uv, $d_{(\bmod r)}(g(u),g(v)) \geq 1$.(see [17, 18].)

The minimum r such that (G,σ) is circular r-colorable is called the circular chromatic number of a signed graph. We note it $\chi_c(G,\sigma) = \inf\{r: (G,\sigma) \text{is circular r-colorable}\}$

In [17], the circular chromatic number for graphs is a refinement of the usual chromatic number. This work introduces the circular chromatic number for planar signed graphs. Recall that, signed graph is a graph G together with a signature $\sigma: E(G) \rightarrow \{+1,-1\}$ which assigns to each edge a sign. So, (G,σ) is said to be *planar* if it can be drawn in the plane without crossing edges.

In his paper, Naserasr et al. in [6] studied, for some special families of signed graphs, the basic properties of the circular coloring and then the upper bounds of the circular chromatic number. Authors consider two graphs: a simple one G and the other one signed $\tilde{G}$. Then they replace every edge uv with a positive edge and a negative edge connecting u,v firstly. Secondly, they establish the relation $\chi_c(\tilde{G}) = 2\chi_c(G)$. Since the result is trivial, they restrict themselves to signed graphs without parallel edges, loops and negative digons. At the end of their search [6], the authors arrive at the clause that there is (G,σ) a planar signed graph of $\chi_c(G) = 4 + \frac{2}{3}$.

According to Gu and Zhu [11] works, this paper continues to reinforce this result by using a method of constructing and prove the following theorem.

Theorem 1 *Let G_r be a planar signed graph. If $r \in [4 + \frac{2}{3}, 6[$, then $\chi_c(G_r) = r$*

In 1997, Moser's research in [2] showed that there is a planar graph G such that $\chi_c(G) = r$ for all rational number $2 \leq r \leq 3$. A few years later, Zhu in [16] has shown that, for every rational number $3 \leq r \leq 4$, there exists a planar graph G such that $\chi_c(G) = r$.

Recently, Gu and Zhu [11] follow the same reasoning. They show that for any rational number $4 \leq \frac{p}{q}$, we can find a simple signed planar graph $G_{p,q}$ such that $\chi_c(G_{p,q}) = \frac{p}{q}$. As in the work of Gu and Zhu [11] on a simple signed planar graph, we follow the same vein with the signed planar graph. Thus, we establish the following corollary.

Corollary 2 *Let (G,σ) be a planar signed graph. $\forall r \in [2,6[$, $\chi_c(G) = r$.*

If all the edges of (G,σ) are positive, then (G,σ) is denoted by G or $(G,+)$). In the continuity of the research carried out in [5, 13] on series-parallel graphs, the circular chromatic number of series-parallel signed graphs is the main object of this article.

In [5], it was proven that for all serial-parallel graph G with at least one edge, $\chi_c(G) \in [2, \frac{8}{3}] \cup \{3\}$. Inversely, there is a serial-parallel graph G so that for all rational $r \in [2, \frac{8}{3}] \cup \{3\}$ we get $\chi_c(G) = r$.

The concept of circular chromatic number of series-parallel signed graphs was studied in [5, 12]. Naserasr et al. in [6] demonstrate that for every seriesparallel signed graph, we get $\chi_c(G,\sigma) \leq 10/3$. In the same vein, they show that for each rational $r \in [2, \frac{10}{3}]$, exists a signed series-parallel graph (G,σ) such that $\chi_c(G,\sigma) = r$.

Every rational number $2 \leq \frac{p}{q} \leq \frac{14}{3}$, the search for a planar signed graph $G_{p,q}$ such that $\chi_c(G_{p,q}) = \frac{p}{q}$ was found recently by Gu and Zhu in [11]. It is known that there are simple signed planar graphs of $\chi_c(G,\sigma) = \frac{14}{3}$. For all signed planar graph (G,σ), we get $\chi_c(G,\sigma) < 6$.

Looking at the results obtained in [5] , [12] and [11] , we observe that there is a significant gap between the upper and lower bounds of the highest circular chromatic number of signed simple planar graphs. In this study, this gap should be reduced. Hence, we have this question:

Question 1: What would be, in the case of a planar signed graph, the highest circular chromatic number?

2. The building method

In this section, we present for a new signed planar graph (G,σ), a construction method such that for $r \in [4 + \frac{2}{3}, 6[$ we obtain $\chi_c(G_r) = r$ The following definition of a serial-parallel signed graph is given by Gu and Zhu in [11].

Definition 1 [(Gu and Zhu, 2022)] Given a signed graph (G,σ) with two distinguished non-adjacent vertices, say s and t, we call (G,σ,s,t) a two-terminal series-parallel signed graph with terminal pair (s,t).

Supposing r is a real positive number, and (G,σ,s,t) a signed two-terminal series-parallel graph. The r-tag set $T_r(G,\sigma,s,t)$ (usually denoted by $T_r(G,\sigma)$, when the terminal vertices are clear from the context) i.e.,

$T_r(G,\sigma) = \{u \in [0,r): \exists\ \phi$: circular r-coloring of (G,σ) such that $\phi(s) = 0$ and $\phi(t) = u\}$

Suppose ϕ is a circular r-coloring of (G,σ), then for any application g and for all vertices u in (G,σ) we have $g(u) = r - \phi(u)$ is also a circular r-coloring of (G,σ).

Note that $T_r(G,\sigma)$ is the symmetric set of colors. To say that $c \in T_r(G,\sigma)$ is like saying that $r - c \in T_r(G,\sigma)$. According to the definition 1, (G,σ) has $r$coloring when and only if $T_r(G,\sigma) \not\models \emptyset$. So, a problem on the determination of $\chi_c(G,\sigma)$ is to determine the smallest rational r such that $T_r(G,\sigma) \not\models \emptyset$ where (G,σ) is a series-parallel signed graph.

Notation 1: $+_{join}$ is denoted a series join and $\|_{join}$ the parallel join.

Lemma 3

1. *Suppose* $(G,\sigma) = (G_1,\sigma_1) +_{join} (G_2,\sigma_2)$, *then* $T_r(G,\sigma) = T_r(G_1,\sigma_1) + T_r(G_2,\sigma_2)$.
2. *Suppose* $(G,\sigma) = (G_1,\sigma_1)\|_{join}(G_2,\sigma_2)$, *then* $T_r(G,\sigma) = T_r(G_1,\sigma_1) \cap T_r(G_2,\sigma_2)$.

Consider (G_1,σ_1,s_1,t_1) and (G_2,σ_2,s_2,t_2) two serial-parallel signed graphs with two terminals having the vertex disjoint.

A series combination of (G_1,σ_1,s_1,t_1) and (G_2,σ_2,s_2,t_2) is a two-terminal series-parallel signed graph. It is obtained from $G_1 \cup G_2$ by identifying t_1 with s_2, and choosing (s_1,t_2) as the new terminal pair.

A parallel combination of (G_1,σ_1,s_1,t_1) and (G_2,σ_2,s_2,t_2) is a two-terminal series-parallel signed graph. It is obtained from $G_1 \cup G_2$ by identifying s_1 with s_2 into a new vertex s, identifying t_1 with t_2 into a new vertex t, and choosing (s,t) as the new terminal pair.

We denote by G_n^{even} (respectively, G_n^{odd}) a signed graph of length n with an even number of positive edges (respectively, an odd number of positive edges). The following lemma is easy and was proved in [12].

Lemma 4 *If* $\forall n \in \mathbb{N}.\ \forall r \in [\frac{14}{3}, \frac{14n}{3n-3}[$,

$$T_r(G_n^{odd}) = [\frac{3n - 3(n-1)r}{14}, \frac{3(n+1)r}{14 - 3n}]$$

$$T_r(G_n^{even}) = [\frac{3n - (3n-14)r}{14}, \frac{3nk}{14 - 3n}]$$

For $r > \frac{14n}{3n-3}$, $T_r(G_n^{odd}) = T_r(G_n^{even}) = C_r$

If (G,σ,s,t) is a signed series-parallel graph, then the odd distance (respectively, the even distance) of (G,σ), denoted $d_{odd}(G,\sigma)$ (respectively, $d_{even}(G,\sigma)$, is the distance of the shortest path $s - t$ in (G,σ) with an odd number of positive edges (respectively, with an even number of positive edges).

The set $d_{odd}(G,\sigma) = \propto$(respectively, $d_{even}(G,\sigma) = \propto$) if the required path does not exist. The distance of (G,σ), is $d(G,\sigma) = \min\{d_{odd}(G,\sigma), d_{even}(G,\sigma)\}$, which is the length of a shortest $s - t$-path in (G,σ).

A discrete version of the definition of the circular chromatic number of a signed graph is given in the following papers [6, 10–13, 13, 14, 17].

Assume $r = \frac{p}{q}$, where p is even. About circular r-coloring of a signed graph

(G,σ), we do not need to use all points in C^r as colors. Instead, it is sufficient to use points of the form $\frac{i}{q} : 0 \leq i \leq p - 1$ as colors.

By omitting the common denominator, we give a definition of a (p,q)coloring of (G,σ) as a function $\psi: V(G,\sigma) \to Z_p$. Every positive edge ab, $q \le \psi(a) - \psi(b) \le p - q$. Every negative edge ab, either $\frac{p}{2} + q \le \psi(a) - \psi(b)$ or $\frac{p}{2} - q \ge \psi(a) - \psi(b)$.

The circular chromatic number of (G,σ) is $\chi_c(G,\sigma) = \inf\{\frac{p}{q}$: p is even and (G,σ) admits a (p,q)-coloring}.

We define $T_{p,q}(G,\sigma) = \{a \in Z_p$: ∃a (p,q)-coloring ψ of G such that $\psi(s) = 0$, for any signed graph (G,σ). The conclusions about $T_r(G,\sigma)$ have corresponding discrete version. In the following, for the proof of the upper bounds, we use the discrete version $T_{p,q}(G,\sigma)$ of label sets, and for the proof of the lower bounds, we use the real number version $T_r(G,\sigma)$.

Claim 2 below follows easily from the definition and straightforward calculations, and he proof is omitted.

Claim 1 Let p be an even integer. If $J = [\frac{p}{2} - 1, \frac{p}{2} + 1]_p$, since J = 3, the following hold:

if $h \ge \frac{p}{2}$, then $(h-1)\bar{J} + J = Z_p$ and $h\bar{J} = Z_p$

If $h < \frac{p}{2}$, then $(h-1)\bar{J} + J = [\frac{p}{2} - h, \frac{p}{2} + h]_p$ and $h\bar{J} = [-h,+h]_p$

If $J = [\frac{p}{2} - 2, \frac{p}{2} + 2]_p$, then since J = 5, the following hold:

If $h \ge \frac{p}{4}$, then $(h-1)\bar{J} + J = Z_p$ and $h\bar{J} = Z_p$

If $h < \frac{p}{4}$, then $(h-1)\bar{J} + J = [\frac{p}{2} - 2h, \frac{p}{2} + 2h]_p$ and $h\bar{J} = [-2h,+2h]_p$

3. Main Theorem 1: Proof

Before we begin the proof of the Theorem 1, let us keep in mind that whenever we write a fraction in the form $\frac{p}{q}$, unless otherwise stated, supposing that $p,q > 0$ and $(p,q) = 1$. Note that for a $\frac{p}{q}$ where $p,q >1$, and $(p,q) = 1$, there are unique integers $0 < a,a' < p$ and $0 < b,b' < q$ such that $pb - aq = 1$ and $a'q - b'p = 1$.

Without prejudice to generality, the fraction $\frac{a}{b}$ is called the lower parent of $\frac{p}{q}$ and let us denote it by $p_l(\frac{p}{q}) = \frac{a}{b}$. Also the fraction $\frac{a'}{b'}$ the upper parent of $\frac{p}{q}$, and denote it by $p_u(\frac{p}{q}) = \frac{a'}{b'}$. Observe that $p = a + a'$ and $q = b + b'$, $pl(pq) < pq < pu(pq)$.

Moreover, in [4] it has been verified that for any fraction $\frac{p}{q}$: $p_l(p_u(\frac{p}{q})) \le p_l(\frac{p}{q})$ and $p_u(p_l(\frac{p}{q})) \ge p_u(\frac{p}{q})$.

Naserasr et al. in [6] construct a complete graph K_4 which is a signed simple planar graph of (G,σ) such that $\chi_c(K_4) = \frac{14}{3}$. We only need to consider this rational number which is strictly between $\frac{14}{3}$ and 6.

Given a rational number $r = \frac{a}{b}$ such that $\frac{14}{3} < \frac{a}{b} <6$ and $(a,b) = 1$, let

$\frac{\frac{14}{3}}{1} < \frac{a_0}{b_0} < \frac{a_1}{b_1} < \ldots < \frac{a_n}{b_n} < \frac{a}{b}$ be the Farey sequence of $\frac{a}{b}$. For $0 \le i \le n-1$,

each rational number $\frac{a_i}{b_i} < r < \frac{a_{i+1}}{b_{i+1}}$ has numerator $> a_{i+1}$.

So, to construct a signed planar simple graph (G,σ) with circular chromatic number equal to $r = {}^a{}_b$, we recursively construct signed planar simple graph (G_i,σ), for $i = 1,2,\ldots,n$, such that (G_i,σ) has circular chromatic number $\frac{a_i}{b_i}$.

After the construction, we prove by induction on i that each signed planar simple graph (G_i,σ) has circular chromatic number $\frac{a_i}{b_i}$. It is relatively easy to prove that $\chi_c(G_i) \le \frac{a_i}{b_i}$.

Lemma 5 $\forall k \in \mathbb{N}\backslash\{1,2\}$, $\exists(G_k,s,t)$: *two-terminal planar signed graph so that*

$$to\ r \in \{\frac{14}{3}, \min\{\frac{14k-9}{3k-3}, 6\}\},\ T_r(G_k) = [\frac{14}{3} - 3r(k-1), k(r - \frac{14}{3})]$$
$$to\ r < \frac{14}{3},\ T_r(G_k) = \emptyset.$$

Proof Let K_4 be the set of vertices s,t,x,y with all positive edges, except that st is a negative edge. Let K be a copy of K_4. Consider the two-terminal signed graph (H_0,s,t)

which is obtained from K by replacing each of the edges sx and sy with a copy of Q.(see Figure 2)

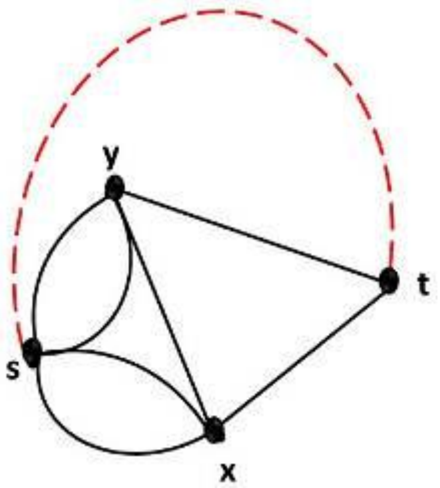

Fig. 2The two-terminal signed graph(G_3,s,t)

Claim 2 $r \in \{\frac{14}{3}, \min\{\frac{14k-9}{3k-3}, 6\}\}$, $T_r(H_0) = [\frac{r}{2}+2, \frac{r}{2} -$ For
$T_r(H_0) = \emptyset$

Proof

2] and if $r < \frac{14}{3}$, Consider that sx and sy are negative edges. Let $K' = H_0[\{s,t,x,y\}]$. We obtain from K_4^+ a copy K' by changing s. Therefore, suppose $r < \frac{14}{3}$, then $T_r(H_0) = \emptyset$.

For $r \in \{\frac{14}{3}, \min\{\frac{14k-9}{3k-3}, 6\}\}$, and $c \in [\frac{r}{2}+2, \frac{r}{2} - 2]$, consider $\psi(s) = 0$ and $\psi(t) = c$, $\psi(x) = \frac{r}{2} - 1$ and $\psi(y) = \frac{r}{2} + 1$. So, the copy of K have a circular r-colouring denoted ψ. As $\psi(x), \psi(y) \in T_r(Q)$, we can extended ψ to a circular r-coloring of H_0.

Therefore, $[\frac{r}{2}+2, \frac{r}{2}-2] \subseteq T_r(H_0)$.

Now, we prove par contrary that $T_r(H_0) \subseteq [\frac{r}{2}+2, \frac{r}{2} - 2]$. Suppose $T_r(H_0) - [\frac{r}{2} + 2, \frac{r}{2} - 2] \neq \emptyset$. Consider that ψ is a circular r-coloring of H_0 so that $\psi(s) = 0$, $\psi(t) = c \notin [\frac{r}{2}+2, \frac{r}{2} - 2]$. Given that st is a negative edge, $c \in [\frac{r}{2}+1, \frac{r}{2}-1] - [\frac{r}{2}+2, \frac{r}{2} - 2]$. By symmetry, we may suppose that $c \in [\frac{r}{2}-2, \frac{r}{2} - 1]$. On the one hand, since the distance from $\psi(x)$ to $\psi(y)$ of c is equal at least 1, $\psi(x), \psi(y) \in [c+1, c-1] \subseteq (\frac{r}{2}-1, \frac{r}{2}-2]$.

At the same vein, $\psi(x), \psi(y) \in T_r(Q) = [3-\frac{r}{2}, \frac{r}{2}-1] \cup [\frac{r}{2}+1, \frac{3r}{2} - 3]$. So, $\psi(x), \psi(y) \in [\frac{r}{2}+1, \frac{3r}{2} - 3]$. Given that $[\frac{r}{2}+1, \frac{3r}{2} - 3]$ is an interval of distance $r - 4 < r - \frac{14}{3} < 1$, we observe that the two vertices x, y are connected with a positive edge. We have a contradiction.

Consider H_1 as the series join of two copies of H_0. Whatever $k \geq 2$, H_k is considered as the series join of H_{k-1} and H_1. Since $r < \frac{14}{3}$, and $T_r(H_0) = \emptyset$, we will consequently have $T_r(H_k) = \emptyset$.

Let us show by induction on $k \in \mathbb{N} \setminus \{1,2\}$. We proof that $r \in$
$\{\frac{14}{3}, \min \frac{14 \; -9}{3k-3}, 6 \quad, T_r(H_k) = [\frac{14}{3} - 3r(k-1), k(r-\frac{14}{3})].\{ \; ^k\}\} \; ^k$

For $k \geq 2$ and $r \in \{\frac{14}{3}, \min\{\frac{14k-9}{3k-3}, 6\}\}$, $T_r(H_{k-1}) = [\frac{14(k-1)}{3} - 3r(k-2), (k-1)(r - \frac{14}{3})]$. Thus, for $r \in \{\frac{14}{3}, \min\{\frac{14k-9}{3k-3}, 6\}\}$,

$T_r(H_k) = T_r(H_{k-1}) + T_r(H_1) = [\frac{14k}{3} - 3r(k-2), k(r-\frac{14}{3})]$.

Since $r <$, the distance of these two intervals is $4^{r(k-\frac{8}{3})} - \frac{14r}{3} < r$. By Lemma 5,

$T_r(P_k) = [\frac{14}{3} - 3r(k-1), k(r-\frac{14}{3})]$.

Lemma 6 *For each rational* $\frac{14}{3} < \frac{p}{q} \leq 5$ *with* $p_l(\frac{p}{q}) < 6$, $\exists(G'_{p,q}, s, t)$ *a two-terminal planar signed graph, if* $p_l(\frac{p}{q}) \geq r < \min\{\frac{3p-14}{3q-3}, 6\}$*, then* $T_r(G'_{p,q}) = [p-1-(q-1)r, qr-p+1]$*. For* $r < p_l(\frac{p}{q})$, $T_r(G'_{p,q}) = \emptyset$.

Proof Suppose $\frac{14}{3} < \frac{p}{q} \leq 5$ and $p_l(\frac{p}{q}) < 6$. consider that $\frac{p}{q} = \frac{14k+3}{3k}$ for k an integer, we have $p_l(\frac{p}{q}) = \frac{14}{3}$. We admit that $G'_{p,q} = H_k$. By the Lemma 5, we have the conculsion. Suppose that $\frac{p}{q} \neq \frac{14k+3}{3k}$ with $k \in \mathbb{N}\backslash\{1,2\}$. As $\frac{p}{q} \in]\frac{14}{3}, 5]$, $\frac{p}{q} \neq \frac{29}{6}$, we have $q \geq 1$. For all $\frac{p'}{q'}$ with $q' < q$, we consider that Lemma 6 is true. Note that $\frac{p}{q} \neq \frac{14k+3}{3k}$ implies that $\frac{a}{b} \neq \frac{14}{3}$. Therefore $\frac{p}{q} \in]\frac{14}{3}, 5]$. Also we have, $p_l(\frac{a}{b}) < \frac{a}{b} = p_l(\frac{p}{q}) <$ 6. By induction hypothesis, $G'_{p,q} = H_k$ is a two-terminal planar signed graph, with $p_l(\frac{a}{b}) \leq r < \min\{\frac{3a-14}{3b-3}, 6\}$, we have $T_r(G'_{a,b}) = [a-1-(b-1)r, br-a+1]$.

Consider P_J as a parallel join of the signed graphs $G'_{a,b}$ and $(K_2,+)$. For $r = \frac{a}{b}$,

$T_r(P_J) = [1, r-1] \cap [a-1-(b-1)r, br-a+1] \subseteq [1, r-1] \cap [r-1, 1] = \emptyset$. For $r \in [\frac{a}{b}, \min\{\frac{3a-14}{3b-3}, 6\}[$, we have $T_r(P_J) = T_r(G'_{a,b}) \cap T_r((K_2,+)) = [a-1-(b-1)r, br-a+1] \cap [1, r-1] = [1, br-a+1] \cup [a-1-(b-1)r, r-1]$.

Consider (S_J,s,t) as a serial join between two copies of P_J. Take x as a known and fixed vertex. Given the disjoint union of (S_J,s,t) and (Q,u,v), and by adding the positive edges vs, vx, vt, us and ut we get (R,s,t). (see Figure 3)

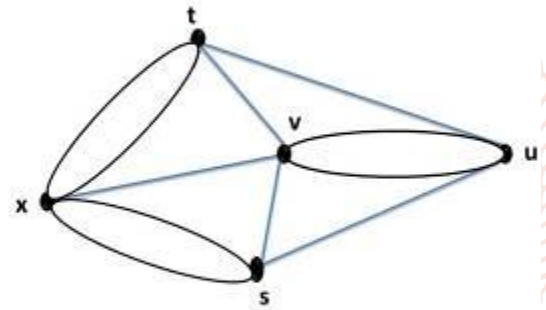

Fig. 3 (R,s,t): Two-terminal signed graph

Claim 3 $\forall r \in [\frac{a}{b}, \min\{\frac{3a-14}{3b-3}, 6\}[$, $T_r = [(1-b)r + a, -a + br]$

Proof The following conditions are satisfied if $\alpha \in T_r(R)$.

1. $\exists s, s' \in T_r(P_J)$ with $\alpha \equiv s + s' (\bmod r)$
2. $\exists w \in [1, r-1] \cap [s+1, s-1] \cap [\alpha+1, \alpha-1]$ and $\exists w' \in [1, r-1] \cap [\alpha+1, \alpha-1]$ with $d_{(\bmod r)}(w, w') \in [3 - \frac{r}{2}, \frac{r}{2} - 1]$.

By induction on α

Case 1 $s \in [1, -a+1+br]$, $s' \in [(1-b)r + a - 1, -1 + r]$ or $s' \in [1, 1-a+br]$, $s \in [(1-b)r + a - 1, -1 + r]$.

Symmetrically, suppose that $s \in [1, -a+1+br]$, $s' \in [(1-b)r + a - 1, -1 + r]$. Consider $w = s+1$, $w' = s$ and $\alpha = s+s' (\bmod r)$. $\alpha \in [a-1-(b-1)r + s, s-1]$. It is easy to check that (1) and (2) are verified. Thus $\alpha \in T_r(R)$. ie $[1, 1-a+br] + [(1-b)r + a - 1, r-1] = [a + (1-b)r, -a + br] \subseteq T_r(R)$

Case 2 $1 \leq s, s' \leq br - a + 1$ or $a - 1 - (b-1)r \leq s, s' \leq r - 1$. In symmetry, we suppose that $1 \leq s, s' \leq br - a + 1$. To solve (1), we have $2 \leq \alpha \leq 2(br - a + 1)$. Since $1 + br - a < \frac{(3a-14)b}{3(b-1)} + 1 - a \leq 2$.

Consequently, if $\alpha > r-2$, $[1, r-1] \cap [s+1, s-1] \cap [\alpha+1, \alpha-1] = \emptyset$ and (2) is not satisfied. If $2 < alphaqr-2$, so $w' \in [1, r-1] \cap [s+1, s-1] \cap [\alpha+1, \alpha-1] = [\alpha+1, r-1]$.

$w' \in [1, r-1] \cap [\alpha+1, \alpha-1] = [1, \alpha-1] \cup [\alpha+1, r-1]$. If $w' \in [\alpha+1, r-1]$, than $d_{(\bmod r)}(w, w') \leq r - 1 - \alpha - 1 = r - \alpha - 2 < r - 4 < 3 - \frac{r}{2}$. If $w' \in [1, \alpha - 1]$, however, then $d_{(\bmod r)}(w, w') \geq 2 > \frac{r}{2} - 1$. Anyway, $d_{(\bmod r)}(w, w') \notin [3 - \frac{r}{2}, \frac{r}{2} - 1]$.

Therefore (2) is not satisfied. Consequently $T_r(R) = [a - (b-1)r, br - a]$ and the

Claim 2 is verified.

Assuming $p_u(\frac{p}{q}) = \frac{5}{1}$, so $G'_{p,q}$ is a series join of R with H_1. $T_r(G'_{p,q}) = T_r(R) + T_r(H_1) = [a-(b-1)r, br-a] + [\frac{14}{3}, r - \frac{14}{3}] = [p-1-(q-1)r, qr-p+1]$. When $p_u(\frac{a'}{b'}) \leq \frac{5}{1}$, then as $p_l(\frac{a'}{b'}) \leq p_l(\frac{p}{q}) < 6$, by the induction hypothesis, there is a planar two-terminal signed graph $G'_{a',b'}$ so that for $p_l(\frac{a'}{b'}) \leq r \leq \min \frac{3a'-14}{3b'-3}, 6\}$, we get $T_r(G'_{a',b'}) = [a'-1-(b'-1)r, b'r-a'+1]$.

Consider $G'_{p,q}$ to be the series join of R and of $G'_{a',b'}$. For $\frac{a}{b} \leq r \leq \min\{\frac{3p-14}{3q-3}, 6\}$, we get $T_r G'_{p,q} = T_r(R) + T_r(G'_{a',b}) = [a-(b-1)r, br-a] + [a'-1-(b'-1)r, b'r-a'+1] =$

$[p-1-(q-1)r, qr-p+1]$. This finishes of the proof of the Lemma 6.

Proof of Main Theorem 1

Assume that $\frac{14}{3} < r < 6$. In view of Lemma 6, a two terminal planar signed graph $(G'_{p,q}, s, t)$ exists with $T_r(G'_{p,q}) = [p-1-(q-1)r, qr-p+1]$. Suppose $G_{p,q}$ is the parallel join of $G'_{p,q}$ and of $(K_2,+)$. Therefore $T_r(G_{p,q}) \notin \emptyset$ iff $\frac{p}{q} \geq r$.

Hence, $\chi_c(G_{p,q}) = \frac{p}{q} \geq \chi_c(G_r) = r$. This complete the Proof of Theorem 1.

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