International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 6 Issue 6, September-October 2022 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

Numerical Simulations of Optical Soliton Propagation under External Forcing

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ABSTRACT

In this study, the time evolution of a single optical soliton and a head-on collision of two well-separated optical solitons both in the presence and absence of fiber loss are consecutively simulated and investigated for the effects of an applied external forcing. The propagation of optical solitons in nonlinear dispersive optical fiber is modelled by one dimensional forced nonlinear Schrödinger (fNLS) equation with the forcing being a time-invariant space-dependent complex Gaussian function. The fNLS equation is approximated by Crank-Nicolson implicit scheme for its one- and two-soliton solutions. The solutions are numerically implemented and simulated in Python 3. In both cases of one and two solitons regardless of whether there is any fiber loss, the external forcing is observed to enforce the generation of a stationary soliton overlapping the forcing function along its spatial coordinate.

KEYWORDS: *forced nonlinear Schrödinger equation, optical soliton, fiber loss, Crank-Nicolson implicit scheme, Gaussian forcing*

How to cite this paper: Soner Tursun | Sharidan Shafie | Ong Chee Tiong "Numerical Simulations of Optical Soliton Propagation under External Forcing" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-6 | Issue-6, October 2022, pp.1940-1948, URL: www.ijtsrd.com/papers/ijtsrd52184.pdf

IJTSRD52184

1. INTRODUCTION

The one-dimensional cubic nonlinear Schrödinger (NLS) equation is given by

$$iu_t + \frac{1}{2}u_{xx} \mp \kappa|u|^2 u = 0 \quad (1)$$

where $u(x,t)$ is a complex wave function, $i^2 = -1$ and κ is an arbitrary nonzero real parameter.

The NLS equation describes electromagnetic phenomena in nonlinear dispersive optical media [1]. It models the propagation of optical pulses in nonlinear optics. Thus, it is used in the construction of fibre-optic communication systems.

In optical fiber, light pulses experience group velocity dispersion (GVD) due to chromatic dispersion and self-phase modulation (SPM) due to the Kerr effect. Optical solitons are formed from the balancing of the effects of GVD and SPM. Optical solitons can propagate undistorted over long distances and they preserve after collisions with other solitons which make them ideal for long-haul fiber optic communication systems.

The existence of optical solitons in optical fiber was first predicted in 1973 as a solution to the nonlinear Schrodinger equation (NLS) [2] and was experimentally verified in 1980 [3]. Since then, the forcing problems for the NLS equation have been less investigated. It is the objective of this study to numerically simulate the effects of a spatial forcing, namely a complex Gaussian function, on the optical solitons of the NLS equation with and without fiber loss.

The NLS equation is an integrable partial differential equation which was first analytically solved by the inverse scattering transform (IST) in 1972 [4]. To obtain forced

Nonlinear Schrödinger (fNLS) Equation, making the NLS equation into a nonhomogeneous partial differential equation by including a forcing term to the right-hand side, turns the NLS equation into a non-integrable system and the number of conservation laws are no longer infinite [5-6]. Furthermore, the equation loses group symmetries due to forcing. The traditional group-theoretical approach can no longer generate analytical solutions which arises a need for a numerical algorithm to approximate the fNLS equation. For that reason, an unconditionally stable finite difference method, Crank-Nicolson implicit scheme [7] is employed to approximate the fNLS equation to find its soliton solutions numerically.

2. Nonlinear Schrödinger Equation

The parameters in equation (1), namely $\frac{1}{2}$ and κ can be almost arbitrarily chosen to set up the suitable variation of the equation for the purpose of a certain study via the scale transformations. The value of the arbitrary real parameter, κ, in equation (1) determines the likelihood of whether any solitons will be generated [8], as such when $\kappa \ll 1$, the nonlinear part of the equation is neglectable and the GVD will overpower the nonlinearity and the solution will just disperse without any nonlinear behaviour, when $\kappa \gg 1$, the SPM will be more evident than dispersion and the solution will 'break' and lastly when $\kappa \approx 1$, then the effects of GVD and SPM will balance each other and there may be found soliton solutions. Hence, to be able to obtain soliton solutions, κ in equation (1) is set to 1. The coefficient before the second linear term is set to 1 as well to obtain one of the fundamental variations of the NLS equations.

On the other hand, due to forcing, a soliton splitting is anticipated. To make any soliton splitting possible and so to obtain bright solitons along with the common effect that the initial profile will evolve into a number of solitons and a dispersive tail [9], the sign before the nonlinear term in equation (1) is set to a plus sign. The variant of the NLS that is used in this study is therefore given by

$$\boldsymbol{iu_t + u_{xx} + |u|^2 u = 0} \quad (2)$$

with the following initial and boundary conditions

$$\boldsymbol{u(x,0) = \psi(x), u(x_L,t) = \alpha, u(x_R,t) = \beta} \ (3)$$

By introducing the forcing term f into the NLS equation, the fNLS equation is settled as

$$\boldsymbol{iu_t + u_{xx} + |u|^2 u = f} \quad (4)$$

where f is a space-wise function, $f=f(x)$.

To represent fibre loss, a linear damping term is introduced to equation (4) to obtain the variant of the fNLS equation with fibre loss as

$$\boldsymbol{iu_t + u_{xx} + |u|^2 u = f - \frac{i\Gamma u}{2}} \quad (5)$$

where $\boldsymbol{-\frac{i\Gamma u}{2}}$ accounts for the linear damping of strength $\boldsymbol{\Gamma > 0}$.

3. Finite Difference Schemes

For discretization, a rectangular mesh on $[\boldsymbol{x_L}, \boldsymbol{x_R}] \times [0, t]$ is constructed.

$$\boldsymbol{\{(x_j, t_n) : x_j = jh, t_n = nk, 0 \le j \le N, 0 \le n \le T\}}$$

where $\boldsymbol{h = \Delta x}$ and $\boldsymbol{k = \Delta t}$ are the sizes of the spatial and temporal steps, respectively.

Crank-Nicolson implicit schemes for equations (2), (4) and (5) are derived via Taylor series expansion. The truncation error of the schemes is of second order in both space and time, $\boldsymbol{\left(O((\Delta t)^2) + O((\Delta x)^2)\right)}$.

3.1. Crank-Nicolson Implicit Scheme for the fNLS Equation

The formulation of Crank-Nicolson implicit scheme for equation (4) is given by,

$$\boldsymbol{i\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{[u_{j+1}^n - 2u_j^n + u_{j-1}^n + u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}]}{2(\Delta x)^2} + \left(|u|_j^n\right)^2\left(\frac{u_j^n + u_j^{n+1}}{2}\right) = f_j} \ (6)$$

where $\boldsymbol{u_j^n}$ is the value of u at the jth grid point at the nth time-step and

$$\boldsymbol{1 \le j \le N-1, 0 \le n \le T-1}$$

with the boundary conditions,

$$\boldsymbol{u_0^n = \alpha, u_N^n = \beta \ 0 \le j \le N}$$

the initial condition for one soliton,

$$\boldsymbol{u_j^0 = \psi(x) \ 0 \le n \le T}$$

and the initial condition for two solitons,

$$\boldsymbol{u_j^0 = \psi(x) + \varphi(x) \ 0 \le n \le T}$$

Rearranging equation (6) for the time steps of n and $n+1$ being on the opposite sides of the equation gives

$$(1+\lambda)\left(u_j^{n+1}\right) - \frac{\lambda}{2}\left[u_{j+1}^{n+1} + u_{j-1}^{n+1}\right] - r\left(|u|_j^n\right)^2 u_j^{n+1} = (1-\lambda)u_j^n + \frac{\lambda}{2}\left[u_{j+1}^n + u_{j-1}^n\right] + r\left[\left(|u|_j^n\right)^2 u_j^n - 2f_j\right] \ (7)$$

where

$$\boldsymbol{\lambda = \frac{i\Delta t}{(\Delta x)^2}, r = \frac{i\Delta t}{2}}$$

Left-hand side of equation (7) can be represented by a tridiagonal matrix M acting on a vector of $\boldsymbol{u^{n+1}}$, namely

$$\boldsymbol{Mu^{n+1}} \ (8)$$

where the matrix M is given by

$$\begin{bmatrix} 1+\frac{\lambda}{2}-\mathrm{r}\left(|u|_j^n\right)^2 & -\frac{\lambda}{2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\frac{\lambda}{2} & 1+\lambda-\mathrm{r}\left(|u|_j^n\right)^2 & -\frac{\lambda}{2} & \ddots & & & & \vdots \\ 0 & -\frac{\lambda}{2} & 1+\lambda-\mathrm{r}\left(|u|_j^n\right)^2 & -\frac{\lambda}{2} & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & -\frac{\lambda}{2} & 1+\lambda-\mathrm{r}\left(|u|_j^n\right)^2 & -\frac{\lambda}{2} & 0 \\ \vdots & & & & \ddots & -\frac{\lambda}{2} & 1+\lambda-\mathrm{r}\left(|u|_j^n\right)^2 & -\frac{\lambda}{2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{\lambda}{2} & 1+\frac{\lambda}{2}-\mathrm{r}\left(|u|_j^n\right)^2 \end{bmatrix} \quad (9)$$

Assigning an iterative function *F* to the right-hand side of equation (7) as

$$\boldsymbol{F_j = (1-\lambda)u_j^n + \frac{\lambda}{2}(u_{j+1}^n + u_{j-1}^n) + r[\left(\left|u_j^n\right|\right)^2 u_j^n - 2f_j]} \quad (10)$$

and by assigning the vectors, $\boldsymbol{u^{n+1}} = \begin{bmatrix} u_0 \\ . \\ . \\ . \\ u_j \\ . \\ . \\ . \\ u_N \end{bmatrix}^{n+1}$ and $\boldsymbol{F} = \begin{bmatrix} F_0 \\ . \\ . \\ . \\ F_j \\ . \\ . \\ . \\ F_N \end{bmatrix}$,

equation (7) can be written in the matrix form as

$$\boldsymbol{Mu^{n+1} = F \; (11)}$$

After implementing the initial and boundary conditions as

$$\boldsymbol{u_j^0 = \psi(x), u_0^n = \alpha, u_N^n = \beta}$$

$$\boldsymbol{F_0 = u_0^n + \frac{\lambda}{2}(u_1^n - u_0^n) + r[(|u_0^n|)^2 u_0^n - 2f_0]}$$

$$\boldsymbol{F_N = u_N^n + \frac{\lambda}{2}(u_{N-1}^n - u_N^n) + r[(|u_N^n|)^2 u_N^n - 2f_N]}$$

equation (11) is solved for $\boldsymbol{u^{n+1}}$ by Gaussian elimination [10] and iteration and direct calculations for each time step.

The finite difference scheme was implemented as a Python 3 algorithm. The program has two main user defined functions. The first function constructs the tridiagonal matrix and the vectors and then determines F_j and solves for u_j using Gaussian elimination [11] while the second function solves the equation for $\boldsymbol{u^{n+1}}$ by direct calculations by time stepping for each space node.

3.2 Crank-Nicolson Implicit Scheme for the fNLS Equation with Fiber Loss

As for equation (5), the finite difference scheme is given by

$$\boldsymbol{i\frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{1}{2(\Delta x)^2}\left[u_{j+1}^n - 2u_j^n + u_{j-1}^n + u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right]}$$
$$\boldsymbol{+\left(|u|_j^n\right)^2\left(\frac{u_j^n+u_j^{n+1}}{2}\right) = f_j - i\frac{\Gamma}{2}\left(\frac{u_j^n+u_j^{n+1}}{2}\right) \; (12)}$$

Separating the terms of *n* and *n+1* gives

$$\boldsymbol{(1+\lambda)u_j^{n+1} + \frac{\lambda}{2}\left[u_{j+1}^{n+1} + u_{j-1}^{n+1}\right] - r\left[\left(|u|_j^n\right)^2 - \sigma\right]u_j^{n+1}}$$
$$\boldsymbol{= (1-\lambda)u_j^n - \frac{\lambda}{2}\left[u_{j+1}^n + u_{j-1}^n\right] - 2rf_j + r\left[\sigma + \left(|u|_j^n\right)^2\right]u_j^n} \quad (13)$$

Where

$$\boldsymbol{\lambda = \frac{i\Delta t}{(\Delta x)^2}, r = \frac{i\Delta t}{2} \text{ and } \sigma = \frac{i\Gamma}{2}}$$

Equation (12) is rearranged to be written in the same matrix form as in equation (11), where *M* is given by

$$\begin{bmatrix}
1+\lambda/2-\mathrm{r}[(|u|_j^n)^2+\sigma] & -\lambda/2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-\lambda/2 & 1+\lambda-\mathrm{r}[(|u|_j^n)^2+\sigma] & -\lambda/2 & \ddots & & & & \vdots \\
0 & \lambda/2 & 1+\lambda-\mathrm{r}[(|u|_j^n)^2+\sigma] & -\lambda/2 & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \lambda/2 & 1+\lambda-\mathrm{r}[(|u|_j^n)^2+\sigma] & -\lambda/2 & 0 \\
\vdots & & & & \ddots & -\lambda/2 & 1+\lambda-\mathrm{r}[(|u|_j^n)^2+\sigma] & -\lambda/2 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -\lambda/2 & 1-\lambda/2-\mathrm{r}[(|u|_j^n)^2+\sigma]
\end{bmatrix}$$

and the iterative functions of vector *F* for equation (11) for the fNLS equation with fibre loss are given by

$$\boldsymbol{F_j = u_j^n + \frac{\lambda}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + r[(|u_j^n|)^2 u_j^n + \sigma - 2f_j]}$$

$$\boldsymbol{F_0 = u_0^n + \frac{\lambda}{2}(u_1^n - u_0^n) + r[(|u_0^n|)^2 u_0^n + \sigma - 2f_0]}$$

$$\boldsymbol{F_N = u_N^n + \frac{\lambda}{2}(u_{N-1}^n - u_N^n) + r[(|u_N^n|)^2 u_N^n + \sigma - 2f_N]}$$

the solutions are obtained by utilizing the same initial and boundary conditions and the Python algorithm as in section (3.1).

4. Simulation Results

The forcing function is chosen to be a complex Gaussian function of space only as

$$\boldsymbol{f = f(x) = i\gamma e^{-(\theta x)^2}} \quad (15)$$

where $\boldsymbol{\gamma}$ **and** $\boldsymbol{\theta}$ are real parameters that determine the amplitude and the width of the forcing, respectively.

Exact soliton solutions of equation (1) for one and two solitons are given in equation (16) and (17), respectively [12].

$$\boldsymbol{u(x,t) = \sqrt{2w}\, e^{i\left[\frac{1}{2}\left(cx-\left(\frac{1}{4}c^2-w\right)t\right)\right]} \operatorname{sech}[\sqrt{w}(x-ct)]} \quad (16)$$

$$\boldsymbol{u(x,t) = \sqrt{2w_1}\, e^{i\left[\frac{1}{2}\left(c_1x_1-\left(\frac{1}{4}c_1{}^2-w_1\right)t\right)\right]} \operatorname{sech}[\sqrt{w_1}(x_1-c_1t)] + \sqrt{2w_2}\, e^{i\left[\frac{1}{2}\left(c_2x_2-\left(\frac{1}{4}c_2{}^2-w_2\right)t\right)\right]} \operatorname{sech}[\sqrt{w_2}(x_2-c_2t)]} \quad (17)$$

The solutions in equation (16) and (17) represent optical solitons that travel at a speed of c_m whose amplitude are determined by the real parameters w_m, where m=1, 2. The solutions will be used to set the initial profiles for the numerical solutions, namely $\boldsymbol{\psi(x)}$ **and** $\boldsymbol{\varphi(x)}$.

The boundary conditions, $\boldsymbol{u(x_L,t) = \alpha, u(x_R,t) = \beta}$, are chosen in a way that $\boldsymbol{\alpha = \beta = 0}$ and x_L *and* x_R are large enough so that the solitons never get close to the boundaries during the computer runs.

4.1. Simulation Results for Single Optical Soliton

One-soliton solution of the fNLS equation without fibre loss, equation (4), is simulated by taking the parameters as

$$\boldsymbol{x_L = -10, x_R = 60, \mathrm{h} = 0.0875, \mathrm{k} = 0.0500,}$$

$$\boldsymbol{\mathrm{w} = 1, \mathrm{c} = 4, \gamma = 0.3, \theta = 7} \quad (18)$$

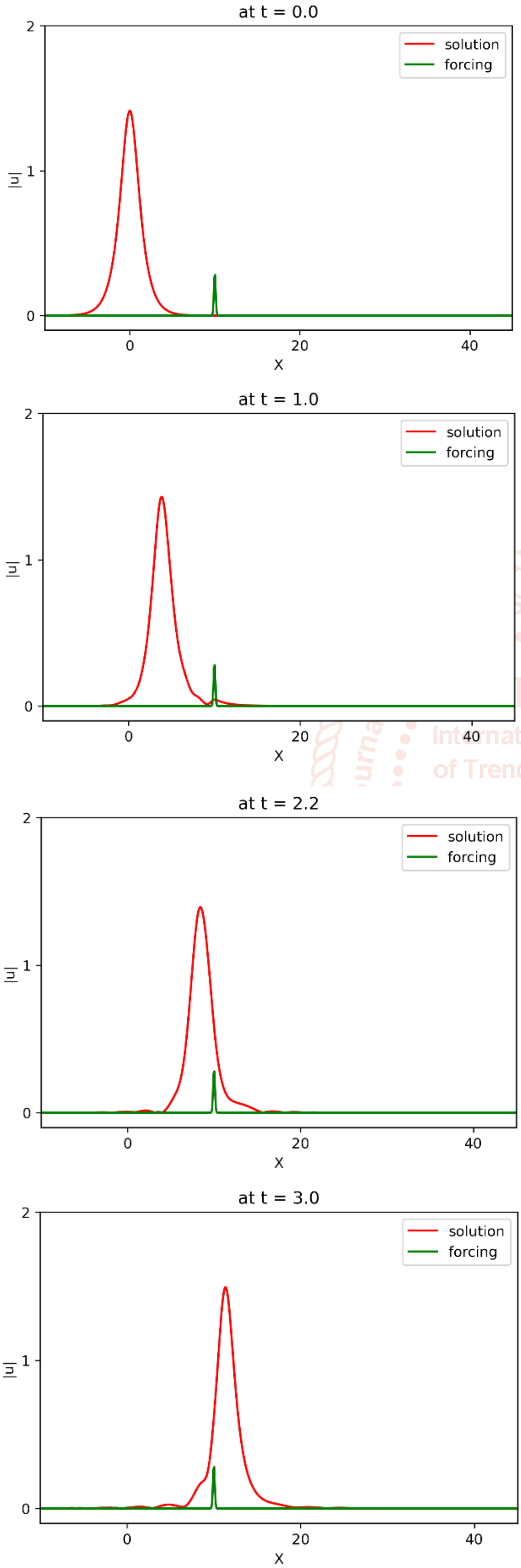
the initial condition as

$$\boldsymbol{\psi(x) = \sqrt{2}e^{2ix}\operatorname{sech}(x)}$$

and so, the forcing function as

$$\boldsymbol{f = f(x) = 0.3ie^{-(7x)^2}} \quad (19)$$

The simulations obtained are given in Figure 1.



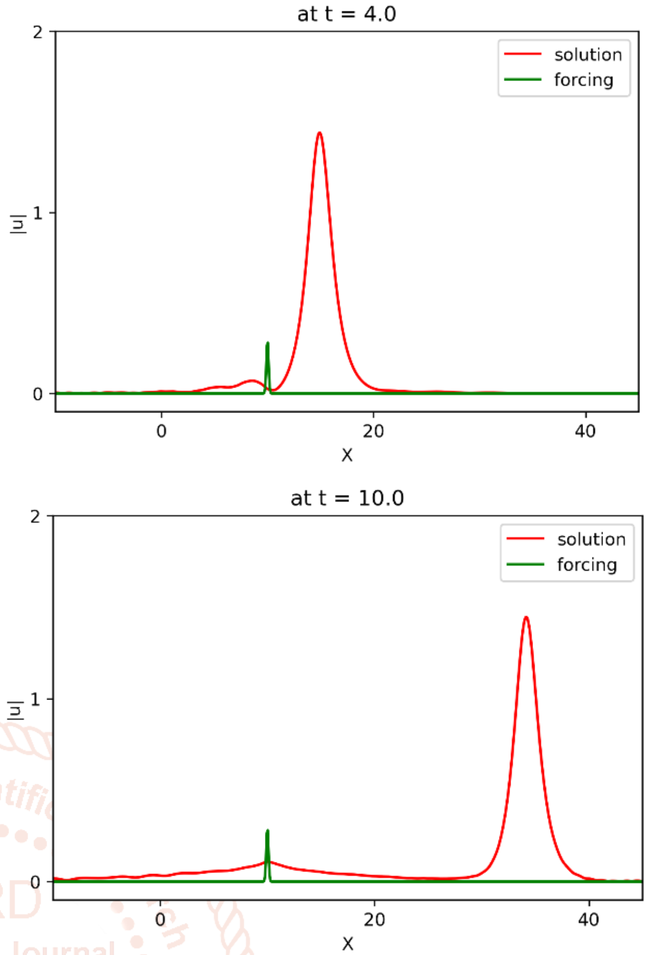
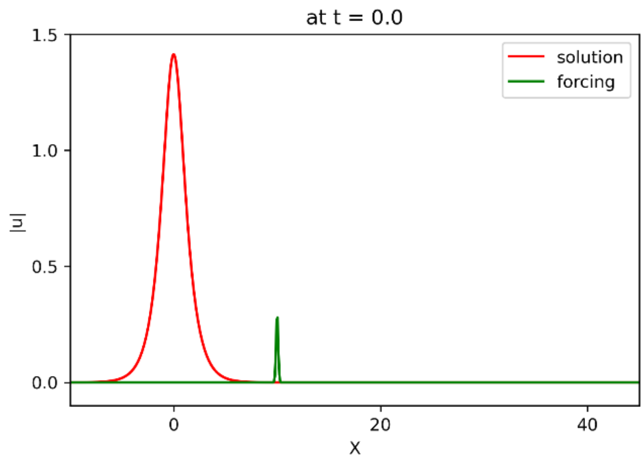
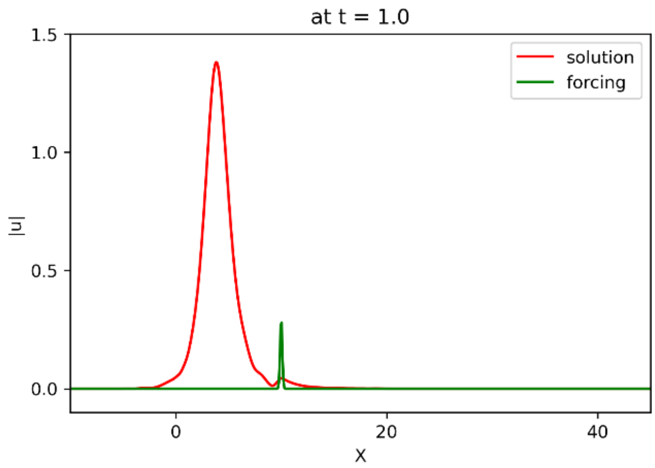


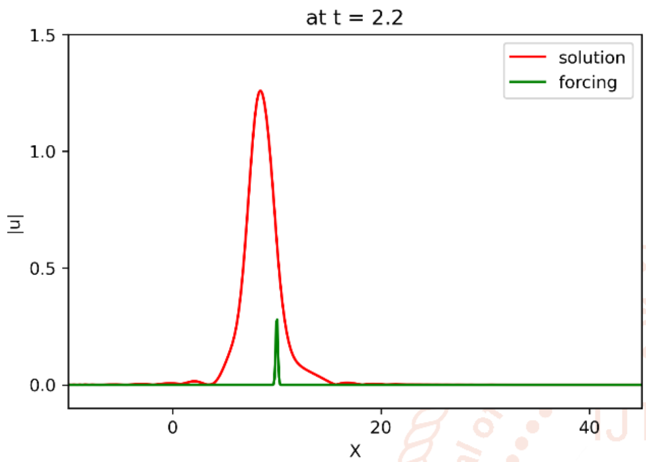
Figure 1: Single soliton of the fNLS equation without fibre loss

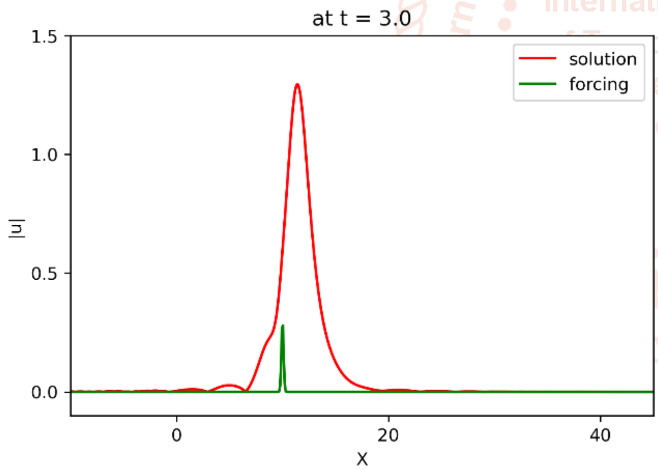
One-soliton solution of the fNLS equation under fibre loss is simulated by introducing a damping factor of $\boldsymbol{\Gamma} = \mathbf{0.05}$ to equation (5). All the other parameters and the initial condition are kept the same as in equations (18).

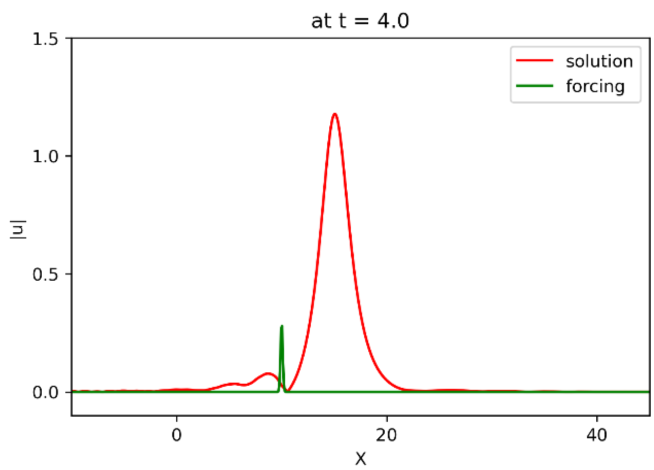
The simulations obtained are given in Figure 2.











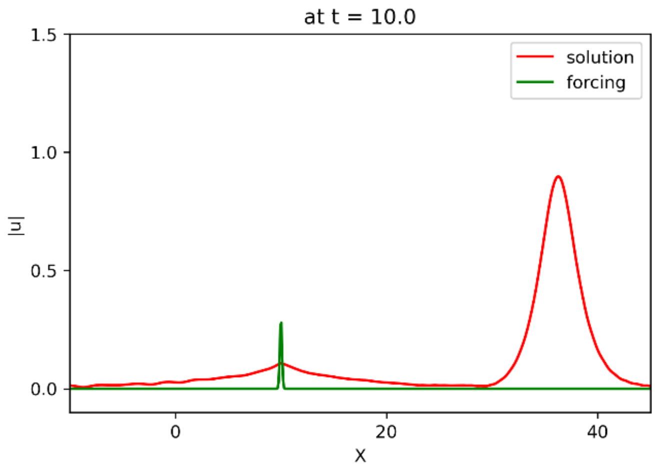


Figure 2: Single soliton of the fNLS equation with fibre loss

4.2. Results for Collision of Two Optical Solitons

The collision of two optical solitons of the fNLS equation that are approaching each other from opposite directions under no fibre loss is simulated with the parameters taken as

$$\boldsymbol{x_L = -65, x_R = 65, \mathrm{h} = 0.0875, \mathrm{k} = 0.0500, w_1 = w_2 = 1, c_1 = 3, c_2 = -3\ \gamma = 0.3, \theta = 5} \quad \textbf{(20)}$$

the initial condition *g(x)* as

$$\boldsymbol{g(x) = \psi(x) + \varphi(x) = \sqrt{2}\left[e^{(1.5ix)}\,\mathrm{sech}(x) + e^{(-1.5ix)}\,\mathrm{sech}(x + 11.4)\right]} \quad \textbf{(21)}$$

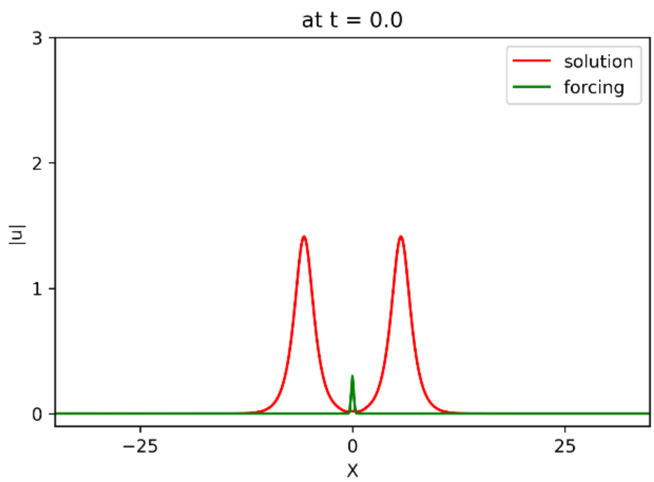
where

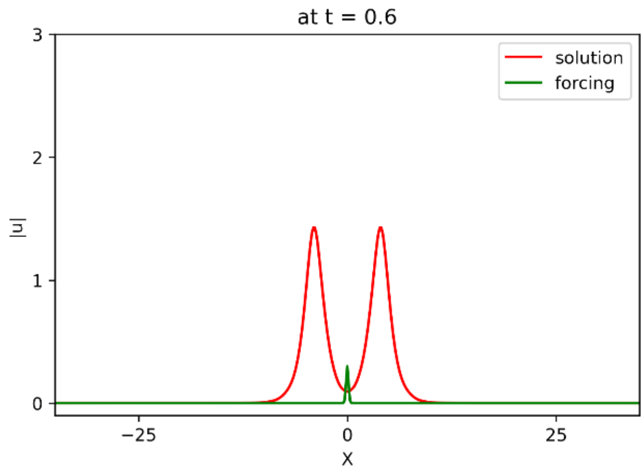
$$\boldsymbol{\psi(x) = \sqrt{2}e^{(1.5ix)}\mathrm{sech}(x)\varphi(x) = \sqrt{2}e^{(-1.5ix)}\mathrm{sech}(x + 11.4)}$$

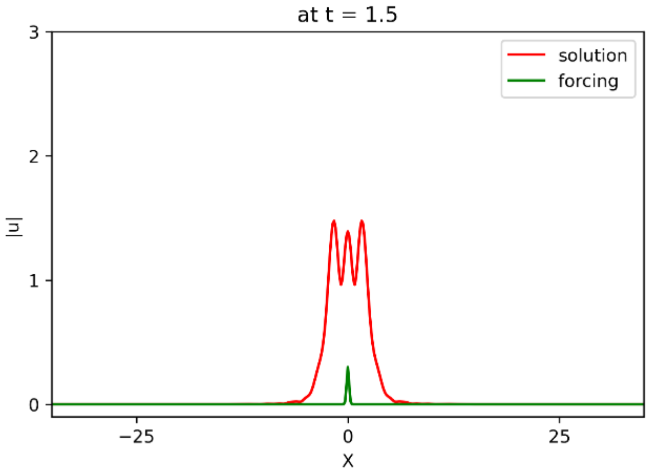
and so, the forcing function as

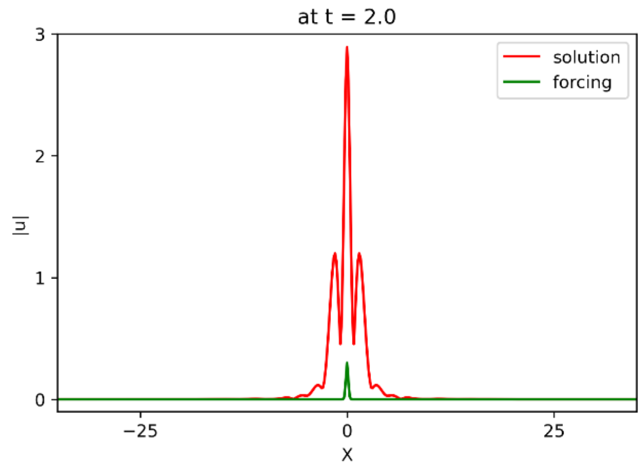
$$\boldsymbol{f = f(x) = 0.3ie^{-(5x)^2}} \quad \textbf{(22)}$$

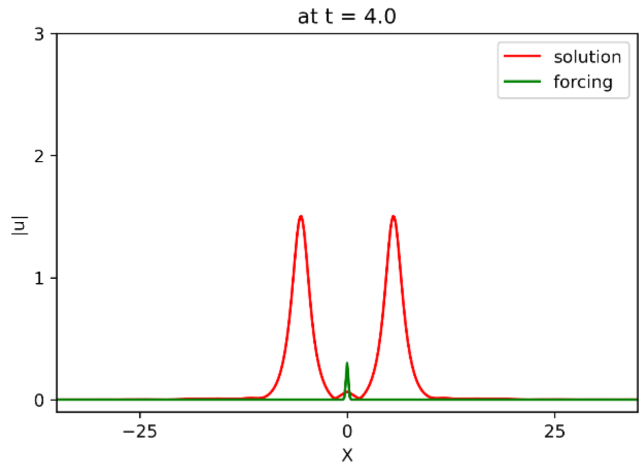
The simulations obtained are given in Figure 3.











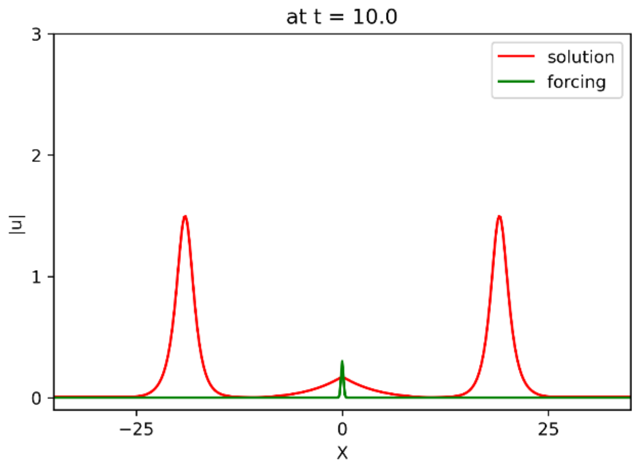
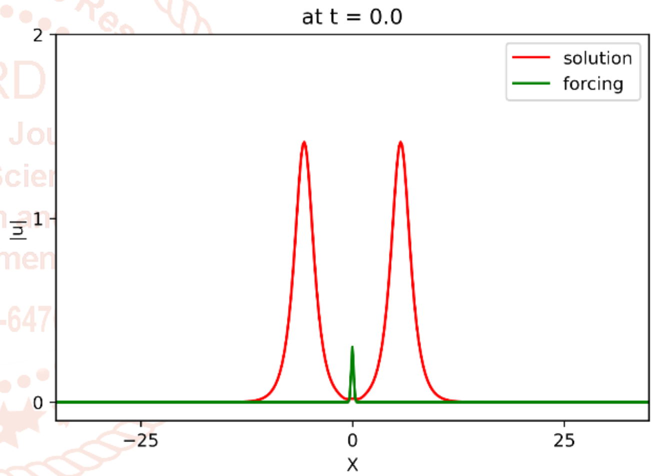
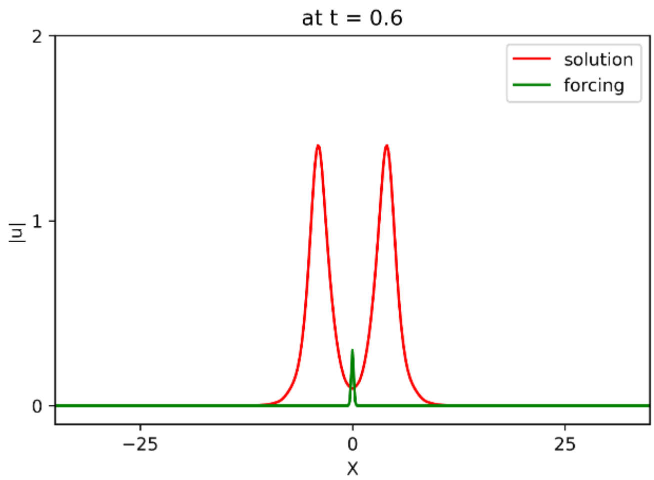


Figure 3: Collision of two solitons of the fNLS equation without fibre loss

The collision of two optical solitons of the fNLS equation under fibre loss is simulated by keeping all the parameters the same as in equations (20) but adding a fibre loss factor of $\boldsymbol{\Gamma = 0.05}$ to equation (5).

The simulations obtained are given in Figure 4.





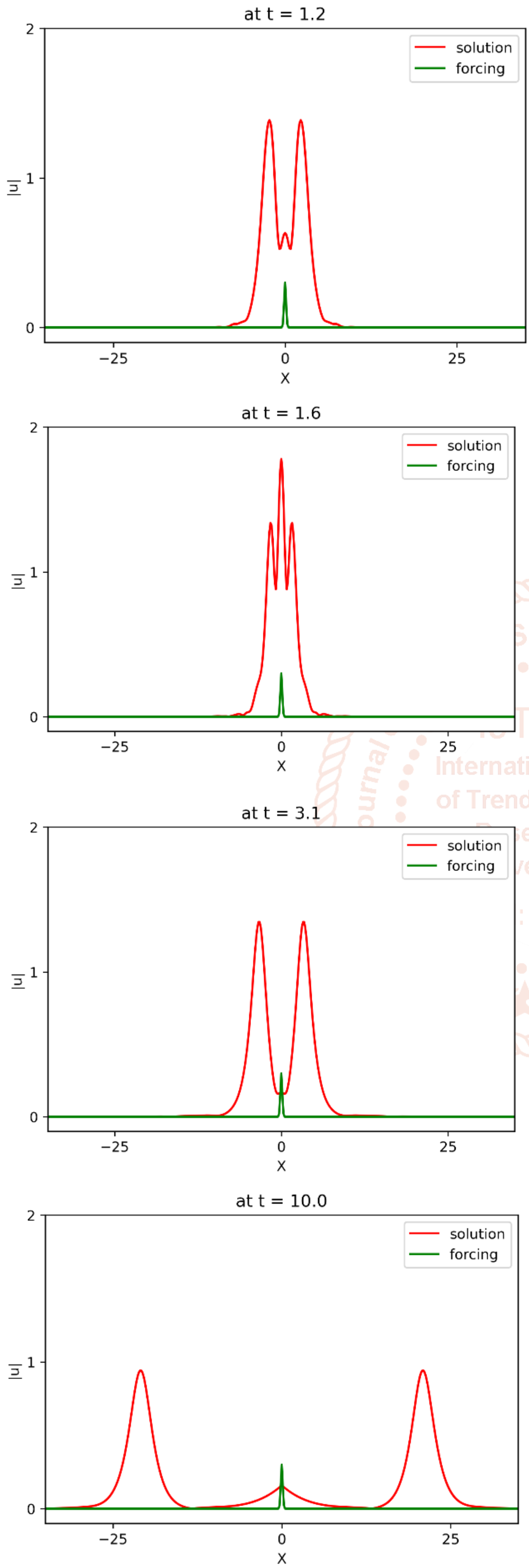


Figure 4: Collision of two solitons of the fNLS equation with fibre loss

5. Conclusions

In this study we investigated the numerical simulations of the propagation of optical solitons in optical fibre under an external spatial forcing. We performed simulations of two sample scenarios to observe the effects of the forcing on bright optical solitons. Firstly, we have focussed on the propagation of a single optical soliton and we continued with the simulation of a head-on collision of two identical solitons under the same sample forcing of a complex Gaussian function. We have repeated the simulations under fibre loss to broaden our investigation.

For the single soliton simulations, we chose our initial profile that is not given by a superposition of multiple solitons. We observed that forcing made the initial profile evolved into two separated solitons one being stationary at the same x-coordinate of the peak of the forcing, and the other one travelling in the same direction of the initial soliton and a dispersive tail travelling in the opposite direction (Figure 1). The moving soliton was narrower than the initial profile while the stationary soliton was wider and shorter. When the fibre loss introduced, both of the resultant solitons have lost power and so their amplitudes decreased considerably (Figure 2).

As the two solitons collided, they underwent a fairly complicated interaction, whereby a tall spike was formed. The process was then reversed, whereby the solitons separated by generating a stationary soliton in the middle where the forcing was applied (Figure 3). Interestingly, unlike the single soliton case, a new soliton wasn’t created from each initial soliton, instead, there was only one soliton generated. The newly generated soliton was not a superposition of two different solitons generated from each initial soliton. We can prove that in two ways, firstly, we would expect the amplitude of the stationary soliton to be doubled if it was a superposition of two new solitons but it is not any larger than the amplitude of the stationary soliton generated in the single soliton case. And if it was a superposition of two small solitons that were split from the initial solitons, we would expect to see two peaks instead of one because of the nonlinearity.

When the fibre loss was introduced, the stationary and the moving solitons had smaller amplitudes over time as a consequence of loss of energy and momentum (power/amplitude) (Figure 4).

Additionally, the change in the amplitude of the optical solitons of both the NLS and fNLS equations as the solitons evolve over time is governed by the sum of the effects of the two terms, namely the nonlinear term which causes breaking

of and the dispersive term which causes widening and spreading of the solitons [13]. It was concluded that a spatial forcing regardless of whether there's fibre loss, will not hinder the balance between SPM and GVD, otherwise they would vanish in time.

The simulated result of the external spatial forcing generating a stationary solution in optical fibre may presumably find a use in a future technology where solitons are needed to be positionally restricted.

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