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Nearly Normal Topological Spaces of the First Kind and the Second Kind

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ABSTRACT

In this paper two new generalizations of normal spaces have been defined and studied. The spaces in these classes have been termed nearly normal topological spaces of the first kind and the second kind respectively.

KEYWORDS: *Normal topological spaces, open set, closed set, quotient spaces, compact Hausdorff spaces, product spaces, metric spaces.*

Mathematics Subject Classification: 54D10, 54D15, 54A05, 54C08

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1. INTRODUCTION

This is the fifth in a series of our papers. The first, the second, the third and the fourth such papers has appeared in 2018([12],[13],[14]) and in 2020([15]). A number of generalizations of normal topological spaces have been defined and studied earlier. p-normal, γ-normal, mildly γ-normal, almost normal and mildly normal spaces ([7], [8], [9], [10], [11]) are several examples of some of these.

In this paper we have defined two new generalizations of normal spaces. These have been called nearly normal topological spaces of the first kind and the second kind. We have provided examples and established many properties of such spaces.

We have used the terminology and definitions of text books of S. Majumdar and N. Akhter [1], Munkres [2], Dugundji [3], Simmons [4], Kelley [5] and Hocking-Young [6].

We now define nearly normal spaces of the first kind and proceed to study them.

2. Nearly Normal Spaces of the First Kind

Definition 2.1: A topological space X will be called **nearly normal of the first kind (n. n. f. k)** if there exists a nontrivial closed set F_0 in X such that, for each nontrivial closed set F in X which is disjoint from F_0, F_0 and F can be separated by disjoint open sets in X. This space will be denoted by **(X, F_0).**

Theorem 2.1: Every normal space is nearly normal space of the first kind but the converse is not true in general.

Proof: Let X be a normal space. Let F_0 be a closed set in X such that, for every closed set H in X such that $F_0 \cap H = \phi$. Now, since X is normal, there exist disjoint open sets G_1, G_2 in X such that $F_0 \subseteq G_1$ and $H \subseteq G_2$. Therefore X is nearly normal space of the first kind.

To see that the converse is always not true,

Let

$$X = \mathbb{R}, \Im = \left\{ \begin{matrix} \mathbb{R}, \emptyset, (1,2), (1,2)^c, (2,3), \\ (2,3)^c, (2,4), (2,4)^c, (2,7), (2,7)^c, (4,5)^c \end{matrix} \right\}$$

Let $F_0 = (1,2)$. Clearly F_0 is closed. $(1,2)^c, (2,3), (2,4), (4,5), (2,7)$ are nontrivial closed sets in X. F_0 can be separated from each of them by open sets, but $(2,4) \cap (4,5)$ and $(2,3) \cap (4,5)$ are disjoint closed sets which can't be separated by disjoint open sets. **Hence (X, F_0) is n. n. f. k. but not normal.**

[Many such examples can be easily constructed.]

Theorem 2.2: A topological space X is nearly normal space of the first kind if and only if there is a nontrivial closed set F_0 in X such that, for every nontrivial closed set F in X which are disjoint from F_0 and an open set G such that $F_0 \subseteq G \subseteq \overline{G} \subseteq (F)^c$.

Proof: First, suppose that X is nearly normal space of the first kind. Then there is a nontrivial closed set F_0 in X such that, for every nontrivial closed set F in X such that $F_0 \cap F = \phi$ and there are open sets G, H in X such that $F_0 \subseteq G$ and $F \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq (F)^c$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq (F)^c$. Thus, $F_0 \subseteq G \subseteq \overline{G} \subseteq (F)^c$.

Conversely, suppose that there is a nontrivial closed set F_0 in X such that, for every nontrivial closed set F in X which are disjoint from F_0 and an open set G such that $F_0 \subseteq G \subseteq \overline{G} \subseteq (F)^c$. Here $F_0 \subseteq G$ and $F \subseteq \overline{G}^c$. Let $\overline{G}^c = H$. Then H is open, $F \subseteq H$ and $G \cap H = \phi$. Hence X is nearly normal space of the first kind.

Theorem 2.3: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i \in I} X_i$ be the product space. If X_i is nearly normal of the first kind, for each i, then X is nearly normal of the first kind.

Proof: Since each X_i is nearly normal of the first kind, there exists, for each $i \in I$, a nontrivial closed set F_i of X_i such that for each nontrivial closed set H_i in X_i with $F_i \cap H_i = \phi$, there are open sets U_i, V_i in X_i such that $F_i \subseteq U_i$, $H_i \subseteq V_i$ and $U_i \cap V_i = \phi$ ……………………………(1)

Let F= $\prod_{i \in I} F_i$. Then F is a nontrivial closed in X. Let K be a nontrivial closed subset of X such that $F \cap K = \phi$. Let, for each $i \in I, \pi_i(K) = K_i$ where $\pi_i : X \to X_i$ is the projection map. Then K_i is nontrivial closed in X_i. By (1), there are open sets W_i, W_i' in X_i such that $F_i \subseteq W_i$, $K_i \subseteq W_i'$. Let $W = \prod_{i \in I} W_i$, $W' = \prod_{i \in I} W_i'$. Then $F \subseteq W$, $K \subseteq W'$ and $W \cap W' = \phi$. Therefore, X is nearly normal space of the first kind.

Theorem 2.4: Every open and one-one image of a nearly normal space of the first kind is nearly normal space of the first kind.

Proof: Let X be a nearly normal space of the first kind and Y a topological space and let $f : X \to Y$ be an open and onto mapping. Since X is nearly normal space of the first kind, there is a nontrivial closed set F in X such that, for every nontrivial closed set H in X such that $F \cap H = \phi$, there are open sets U, V in X such that $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. Since f is open, $f(F^c)$ and $f(H^c)$ are open in Y. So $(f(F^c))^c$ and $(f(H^c))^c$ are closed in Y.

Now, $F^c \cup H^c$ =X and so $f(F^c \cup H^c)$ =Y, i.e., $f(F^c) \cup f(H^c)$=Y.

Hence $(f(F^c))^c \cap (f(H^c))^c = \Phi$.

Let $y_0 \in (f(F^c))^c$. Then $y_0 \notin f(F^c)$ i.e., there exists $x_0 \in F^c$, $f(x_0) \neq y_0$. Hence $x_1 \in F$ such that $f(x_1) = y_0$, since f is onto. Thus $y_0 \in f(F)$. Hence $(f(F^c))^c \subseteq f(F)$. Similarly, $(f(H^c))^c \subseteq f(H)$.

Now, $f(F) \subseteq f(U)$, $f(H) \subseteq f(V)$, f being open and one-one, $f(U)$, $f(V)$ are open and disjoint in Y. Thus for a nontrivial closed set $(f(F^c))^c$ in Y such that, for every nontrivial closed sets $(f(H^c))^c$ in Y such that $(f(F^c))^c \cap (f(H^c))^c = \phi$, there are open sets $f(U)$, $f(V)$ in Y such that $(f(F^c))^c \subseteq f(U)$, $(f(H^c))^c \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Hence Y is nearly normal space of the first kind.

Corollary 2.1: Every quotient space of a nearly normal space of the first kind is nearly normal space of the first kind.

Proof: Let X be a nearly normal space of the first kind and R is an equivalence relation on X. Since the projection map p:X$\to \frac{X}{R}$ is open and onto, the corollary then follows from the above Theorem 2.4.

Theorem 2.5: Let X be a nearly normal space of the first kind and Y is a subspace of X. Then Y is a nearly normal space of the first kind.

Proof: Since X is nearly normal space of the first kind, there is a nontrivial closed set F in X such that, for every nontrivial closed set H in X such that $F \cap H = \phi$, there are open sets U, V in X such that $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. Let $F' = Y \cap F$ and $H' = Y \cap H$. Then for a nontrivial closed set F' in Y such that, for every nontrivial closed set H' in Y such that $F' \cap H' = \phi$. Also let $U' = Y \cap U$, $V' = Y \cap V$. Then U', V' are open sets in Y and $U' \cap V' = \phi$ and $F' \subseteq U'$, $H' \subseteq V'$. Hence Y is nearly normal space of the first kind.

Remark 2.1: The corresponding theorem does not hold for normal spaces. The validity of the proof in Theorem 2.5 above depends on the separablity of a particular pair of disjoint closed spaces by disjoint open spaces (See Ex. of Munkres[2]).

Comment 2.1: A continuous image of a nearly regular space of the first kind (nearly normal space of the first kind) need not be nearly regular space of the first kind (nearly normal space of the first kind).

For if (X, T_1) is a nearly regular space of the first kind (nearly normal space of the first kind) and (X, T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X, T_2) is not nearly regular space of the first kind (nearly normal space of the first kind).

Theorem 2.6: Each compact Hausdorff space is nearly normal space of the first kind.

Proof: Let X be a compact Hausdorff space and let for a nontrivial closed subset A, there is a nontrivial closed subset B in X which is disjoint from A. Let $x \in A$ and $y \in B$. Then $x \neq y$. Since X is Hausdorff, there exist disjoint open sets G_y and H_y such that $x \in G_y$ and $y \in G_y$. Obviously $\{H_y : y \in B\}$ is an open cover of B.

Since B is a closed subset of X, B is compact. So there exists a finite subcover $\{H_{y_1}, H_{y_2}, \dots, H_{y_m}\}$ of B. Let $H_x = H_{y_1} \cup H_{y_2} \cup \dots \cup H_{y_m}$ and $G_x = G_{y_1} \cap G_{y_2} \cap \dots \cap G_{y_m}$. Then $B \subseteq H_x$, $x \in G_x$ and $H_x \cap G_x = \phi$ i.e., X is nearly regular space of the first kind. So for each $x \in A$, there exist two disjoint open sets G_x and H_x of X such that $x \in G_x$ and $B \subseteq H_x$. Hence $\{G_x : x \in A\}$ is an open cover of A. Since A is a closed subset of X, A is compact. So there exists a finite subcover $\{G_{x_1}, G_{x_2}, \dots, G_{x_n}\}$ of this cover A. Let $G = G_{x_1} \cup G_{x_2} \cup \dots \cup G_{x_n}$ and $H = H_{x_1} \cap H_{x_2} \cap \dots \cap H_{x_n}$. Then G, H are open sets of X and $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$. Hence the proof.

Remark 2.2: It follows from the above proof that every compact Hausdorff space is nearly regular space of the first kind.

Theorem 2.7: Every locally compact Hausdorff space is nearly regular space of the first kind.

Proof: Let X be a locally compact Hausdorff space. Then there exists one point compactification X_∞ of X. Then, X_∞ is Hausdorff and compact. According to the above Remark 2.2, X_∞ is nearly regular space of first kind. Again, according to Theorem 2.5, as a subspace of X_∞, X is nearly regular space of the first kind.

Theorem 2.8: Let X be a T_1- space. Then X is nearly normal space of the first kind if and only if X is nearly regular space of the first kind.

Proof: First, suppose that X be a nearly normal space of the first kind. Let x be a point in X and let F_0 be a nontrivial closed subset of X such that $x \notin F_0$. Since X is T_1- space, {x} is closed subset of X. We have $\{x\} \cap F_0 = \phi$. Since X is nearly normal space of the first kind, there are open sets G and H such that $\{x\} \subseteq G$, $F_0 \subseteq H$, $G \cap H = \phi$ i.e., $x \in G$, $F_0 \subseteq H$, $G \cap H = \phi$. Hence X is nearly regular space of the first kind.

Conversely, suppose that X be a nearly regular space of the first kind. Let x be a point in X and let F_0 be a nontrivial closed subset of X such that $x \notin F_0$. Since X is T_1- space, {x} is closed subset of X. We have $\{x\} \cap F_0 = \phi$. Since X is nearly regular space of the first kind, there exist open sets G and H such that $x \in G$, $F_0 \subseteq H$, $G \cap H = \phi$ i.e., $\{x\} \subseteq G$, $F_0 \subseteq H$, $G \cap H = \phi$. Hence X is nearly normal space of the first kind.

Theorem 2.9: Every metric space is nearly normal space of the first kind.

Proof: Since every metric space is normal, therefore it is nearly normal space of the first kind.

We now define nearly normal spaces of the second kind and proceed to study them.

3. Nearly Normal Spaces of the Second Kind

Definition 3.1: A topological space X will be called **nearly normal of the second kind (n. n. s. k)** if for each nontrivial closed set F_1, there exists a nontrivial closed set F_2 in X which is disjoint from F_1 such that F_1 and F_2 can be separated by disjoint open sets in X.

Example 3.1: Every n. n. f. k. is n. n. s. k.

[We are to construct an example of an n. n. s. k. space which is not n. n. f. k.]

Theorem 3.1: Every normal space is nearly normal space of the second kind but the converse is not true in general.

Proof: Let X be a normal space. Let F be a closed set in X such that, there exists a closed set H in X such that $F \cap H = \phi$. Now, since X is normal, there exist disjoint open sets G_1, G_2 in X such that $F \subseteq G_1$ and $H \subseteq G_2$. Therefore X is nearly normal space of the second kind.

To see that the converse is always not true,

the proof is most similar to the proof of the last part of Theorem 2.1of **n. n. f. k.**

Theorem 3.2: A topological space X is nearly normal space of the second kind if and only if for each nontrivial closed set F in X such that, there is a nontrivial closed set F_0 in X which is disjoint from F and an open set G such that $F \subseteq G \subseteq \overline{G} \subseteq F_0^{\,c}$.

Proof: First, suppose that X is nearly normal space of the second kind. Then for each nontrivial closed set F in X such that, there is a nontrivial closed set F_0in X such that $F_0 \cap F = \phi$ and there are open sets G, H in X such that $F \subseteq G$ and $F_0 \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F_0^{\,c}$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F_0^{\,c}$. Thus, $F \subseteq G \subseteq \overline{G} \subseteq F_0^{\,c}$.

Conversely, suppose that for each nontrivial closed set F in X such that, there is a nontrivial closed set F_0 in X which is disjoint from F and an open set G such that $F \subseteq G \subseteq \overline{G} \subseteq F_0^{\,c}$. Here $F \subseteq G$ and $F_0 \subseteq \overline{G}^{\,c}$. Let $\overline{G}^{\,c} = H$. Then H is open, $F_0 \subseteq H$ and $G \cap H = \phi$. Hence X is nearly normal space of the second kind.

Theorem 3.3: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i \in I} X_i$ be the product space. If X_i is nearly normal of the second kind, for each i, then X is nearly normal of the second kind.

Proof: Since each X_i is nearly normal of the second kind, for each $i \in I$, for each nontrivial closed set F_i of X_i such that there exists a nontrivial closed set H_i in X_i with $F_i \cap H_i = \phi$, there are open sets U_i, V_i in X_i such that $F_i \subseteq U_i$, $H_i \subseteq V_i$ and $U_i \cap V_i = \phi$ ....(1)

Let F= $\prod_{i \in I} F_i$. Then F is closed in X. Let K be a nontrivial closed subset of X such that $F \cap K = \phi$.Let, for each $i \in I, \pi_i(K) = K_i$ where $\pi_i : X \to X_i$ is the projection map. Then K_i is closed in X_i. By (1), there are open sets W_i, $W_i^{\,/}$ in X_i such that $F_i \subseteq W_i$, $K_i \subseteq W_i^{\,/}$. Let $W = \prod_{i \in I} W_i$, $W^{/} = \prod_{i \in I} W_i^{\,/}$. Then $F \subseteq W$, $K \subseteq W^{/}$ and $W \cap W^{/} = \phi$. Therefore, X is nearly normal space of the second kind.

Theorem 3.4: Every open and one-one image of a nearly normal space of the second kind is nearly normal space of the second kind.

Proof: The proof of the Theorem 3.4 of the above is almost similar to the proof of the Theorem 2.4.

Corollary 3.1: Every quotient space of a nearly normal space of the second kind is nearly normal space of the second kind.

Proof: The proof of the Corollary 3.1is most similar to the proof of the Corollary 2.1.

Theorem 3.5: Let X be a nearly normal space of the second kind and Y is a subspace of X. Then Y is a nearly normal space of the second kind.

Proof: The proof of the Theorem 3.5 follows from the proof of the Theorem 2.5.

Remark 3.1: The corresponding theorem does not hold for normal spaces. The validity of the proof in Theorem 3.5 above depends on the separablity of a particular pair of disjoint closed spaces by disjoint open spaces (See Ex. of Munkres [2]).

Comment 3.1: A continuous image of a nearly regular space of the second kind (nearly normal space of the second kind) need not be nearly regular space of the second kind (nearly normal space of the second kind).

For if (X, T_1) is a nearly regular space of the second kind (nearly normal space of the second kind) and

(X,T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X,T_2) is not nearly regular space of the second kind (nearly normal space of the second kind).

Theorem 3.6: Each compact Hausdorff space is nearly normal space of the second kind.

Proof: The proof of the Theorem 3.6 is most similar to the proof of the Theorem 2.6.

Theorem 3.7: Every locally compact Hausdorff space is nearly regular space of the second kind.

Proof: The proof of the Theorem 3.7 of the above is almost similar to the proof of the Theorem 2.7.

Theorem 3.8: Let X be a T_1- space and x_0 be a point in X. X is nearly normal space of the second kind if and only if X is nearly regular space of the second kind.

Proof: The proof of the Theorem 3.8 is almost similar to the proof of the Theorem 2.8.

Theorem 3.9: Every metric space is nearly normal space of the second kind.

Proof: Since every metric space is normal, therefore, it is nearly normal space of the second kind.

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