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Generalization of Fibonacci Numbers with Binomial Coefficients and Figurate Numbers

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ABSTRACT

Fibonacci Sequence has formed the groundwork of Mathematicians for research in Number Theory. Fibonacci has a profound influence all around us. There are many interesting identities involving Fibonacci. With the invention of Binet's Formula (which is the most fundamental formula for finding nth Fibonacci), further search for generalisation of Fibonacci sequence has almost came to an end but it is interesting to know that Fibonacci numbers can be expressed by Binomial Coefficients. Thus, I tried to convey the process of generalisation of Fibonacci using Binomial coefficients.

KEYWORDS: *Introduction and Application of Fibonacci, Fibonacci Numbers from Pascal's Triangle, Fibonacci Numbers using Binomial Coefficients, Fibonacci Numbers from Figurate Numbers, Some Interesting Sums Involving Fibonacci*

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I. INTRODUCTION

Among the greatest Mathematicians of the Middle Ages Leonardo of Pisa is considered to be one of the greatest Mathematician. He is mainly known for his work on *Fibonacci*. Today, most of us are acquainted with the *Fibonacci* Sequence.

The *Fibonacci* Sequence follows as 1,1,2,3,5,8,13,21…

This sequence has a profound influence around us ranging from the spirals of sunflower, cones of pine tree and breeding of rabbits.

Fibonacci Sequence has a wide application in Probability, Combinatorics, Continued Fractions, Solving Pell's Equation etc.

Fibonacci gives answer to the problems like,
1. Finding the probability of colouring a n-storey house with the precondition that no two adjacent storey has the same colour.
2. Let's suppose you have to calculate the number of ways in which you have to pay multiples of Rs.25 using only Rs.25 and Rs.50.

Fibonacci gives answer to all these problems.

It also plays a very important part in Infinite Continued Fractions and it's very interesting to know that Fibonacci can be also used to generate Pythagorean Triples, and that's too Primitive Pythagorean Triples.

For generating Pythagorean Triples take any four consecutive Fibonacci Numbers from the sequence.

Let's suppose we choose 1,2,3,5.
1. At first double the product of the mean numbers.

(Here, $2(2 \times 3) = 12$; 12 is our first member of the triples.)

2. Next multiply the extreme numbers.(Here, $5 \times 1 = 5$; This 5 is our second element of the triples.)
3. At last, sum of the squares of the mean numbers form our last triple.(Here, $2^2 + 3^2 = 4 + 9 = 13$; This 13 is finally our last triple.)

Finally, our Pythagorean Triple is (12,5,13)
We can check our answer;

$$12^2 + 5^2 = 144 + 25 = 169 = 13^2$$

This can be generalised as following;
F_n denotes the nth Fibonacci element. Our four Fibonacci Elements will be F_n, F_{n+1}, F_{n+2}, F_{n+3}.

(a,b,c) forms our ordered Pythagorean Triples.

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}) = (2\mathrm{F}_{\mathrm{n}+1} + 2\mathrm{F}_{\mathrm{n}+2}, \mathrm{F}_{\mathrm{n}} \times \mathrm{F}_{\mathrm{n}+3}, {\mathrm{F}_{\mathrm{n}+1}}^2 + {\mathrm{F}_{\mathrm{n}+2}}^2)$$

The general term for Fibonacci can be concluded as,
$F_n = F_{n-1} + F_{n-2}$(i.e., each term is formed by summing previous terms).

II. PASCAL'S TRIANGLE

Pascal Triangle was discovered by French Mathematician Blaise Pascal. The Pascal's Triangle is a triangle formed by the coefficients of the binomial expansion.

The triangle looks as follows.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Consider that 0 is placed at the sides of the pascal's triangle. The first rule is that the numbers at the two ends will be formed by summing the top two terms.

All the numbers at the end are 1.The middle elements of Pascal's Triangle is actually formed by the binomial coefficients following the Pascal's Rule.

The first element of Pascal's Triangle is 1 represented by Binomial Coefficient as $\binom{0}{0}$.

So, the first element is $\binom{0}{0}$.

The next terms of the Pascal's Triangle are formed by summing the top binomial coefficients.

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \quad 0 \le k \le n$$

Let's check how it forms.

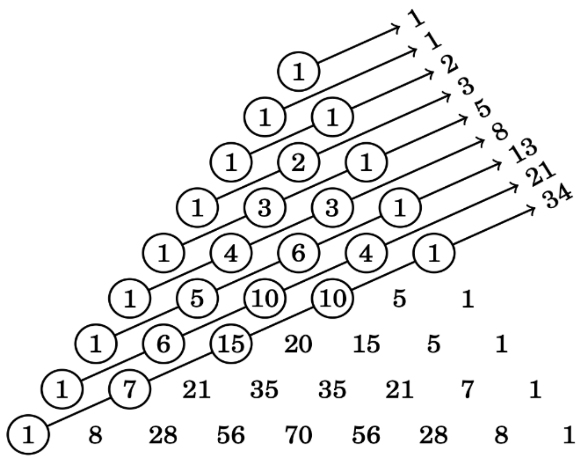
$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

All the terms at the ends are 1 and thus following the Coefficients of Binomial Expansions, the first and the last terms are given a as $\binom{n}{0}$ and $\binom{n}{n}$ respectively where n denotes the row numbers. The middle elements are formed by following the pascal's rule.

III. FIBONACCI NUMBERS FROM PASCAL'S TRIANGLE

It is interesting to know that Fibonacci Numbers can also be derived from the pascal's triangle. This was first observed by British Mathematician Ron Knott. Later this was studied by French Mathematician Francois Edouard Anatole Lucas. The diagonals of a Pascal's Triangle add up to the nth Fibonacci Number. Let's check how it comes.

The diagonals of a Pascal Triangle can be added up in this way to get the Fibonacci Numbers.



Summing up the diagonals in this way can give us the Fibonacci Numbers from Pascal's Triangle.

IV. REPRESENTATION OF FIBONACCI NUMBERS USING BINOMIAL COEFFICIENTS.

Fibonacci Numbers can be derived from summing the diagonals of Pascal's Triangle which in turn is actually formed by Binomial Coefficients.

This paves our path that in some way Fibonacci Numbers can be generalised using Binomial Coefficients.

By the formation of Pascal Triangle using Binomial Coefficients, let relate the Fibonacci Numbers in similar way.

The Fibonacci terms can be represented using the Binomial Coefficients in the following way.

Let *F(n)* denote the Fibonacci Number at *nth* term.

$$F(1) = 1 = \binom{0}{0}$$

$$F(2) = 1 = \binom{1}{0}$$

$$F(3) = 2 = \binom{2}{0} + \binom{1}{1}$$

$$F(4) = 3 = \binom{3}{0} + \binom{2}{1}$$

$$F(5) = 5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2}$$

$$F(6) = 8 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2}$$

$$F(7) = 13 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}$$

$$F(8) = 21 = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$$

The sequence continues in this way.

V. GENERALISATION OF FIBONACCI NUMBERS USING BINOMIAL COEFFICIENTS

For a Fibonacci Number F_n, the representation using summation of Binomial Coefficients follows a specific pattern. This pattern can be generalised as,

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \binom{n-4}{3} + \binom{n-5}{4} + \binom{n-6}{5} + \cdots$$

By a slight observation of the above pattern one can deduce that the sum of the upper and lower terms of the Binomial Coefficients is always $(n-1)$ *for the* F_n *Fibonacci Number.*

Now, we have to generalise this sum in a closed form.

We have to derive a formula for finding the nth term of Fibonacci(F_n). According to the division algorithm, every $n (n \in \mathbb{N})$ is either of the form $2x$ or $2x + 1$. [As when divisible by 2]

Considering the case of $2x + 1$. Adding and subtracting 1 to it we get $2x + 1$.

ALGORITHM - I

For any Fibonacci Number F_n and $n \in P$(clear that $P \subset \mathbb{N}$)where

$$P = \{2x - 1 \mid x \in \mathbb{Z};\ 2x - 1 > 0\}$$

The lower term of the coefficients run from $k = 0$ to $k = \frac{n-1}{2}$. So as to meet our precondition that the sum of upper term and the lower term of the binomial coefficients must be $n - 1$, the upper term must be assigned a value of $(n-1) - k$ to make the sum of the upper and lower terms as $n - 1$.

Therefore, for any Fibonacci Number F_n and $n \in P$ where $P = \{2x - 1 \mid x \in \mathbb{Z};\ 2x - 1 > 0\}$.

The F_n is defined as the sum of coefficients whose upper term is $(n-1) - k$ and the lower term runs from $k = 0$ to $k = \frac{n-1}{2}$.

In a more generalised way this can be written as,

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k}\ \forall\, n \in P \cdots (1)$$

ALGORITHM - II

Our next aim is to generalise the formula for F_n in which $n \notin P$.

It is clear that the set P includes all the possibilities of *n* which is of the form $2x - 1$.

Next, we have to find a formula for *n* which is of the form $2x$.

For any Fibonacci Number F_n and $n \in Q$(clear that $Q \subset \mathbb{N}$) where

$$Q = \{2x \mid x \in \mathbb{Z};\ 2x > 0\}$$

The lower term of the coefficients run from $k = 0$ to $k = \frac{n-2}{2}$. So as to meet our precondition that the sum of upper term and the lower term of the binomial coefficients must be $n - 1$, the upper term must be assigned a value of $(n-1) - k$ to make the sum of the upper and lower terms as $n - 1$.

Therefore, for any Fibonacci Number F_n and $n \in Q$ where $Q = \{2x \mid x \in \mathbb{Z};\ 2x \geq 0\}$. The F_n is defined as the sum of coefficients whose upper term is $(n-1) - k$ and the lower term runs from $k = 0$ to $k = \frac{n-2}{2}$.

In a more generalised way this can be written as,

$$F_n = \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} \ \forall\, n \in Q \cdots (2)$$

Through the equations (1) and (2), all the possibilities of n are covered.

Our final conjecture is given as follows,

$$F_n = \begin{Bmatrix} \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} \ \forall\, n \in P \\ \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} \ \forall\, n \in Q \end{Bmatrix}$$

Let this be considered as a Fi-binomial conjecture.

THEOREM – I :

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} \ \forall\, n \in P$$

PROOF:

The above theorem can easily be proved by Mathematical Induction.

Let define it as a function.

$$f(n) = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} \ \forall\, n \in P$$

We have our set P as,

$$P = \{2x-1 \mid x \in \mathbb{Z};\ 2x-1 > 0\}$$

In, set P it is clear that the set elements differ by 2.

By the definition of Fibonacci we know;

$$f(n+1) = f(n) + f(n-1)$$

BASE PROOF:

$$f(1) = \sum_{k=0}^{0} \binom{1-k-1}{k}$$

$$= \binom{1-0-1}{0} = \binom{0}{0} = 1$$

$$f(3) = \sum_{k=0}^{1} \binom{3-k-1}{k}$$

$$= \binom{2}{0} + \binom{1}{1} = 1 + 1 = 2$$

We have verified our base step.

INDUCTION HYPOTHESIS:

We have,

$$f(n) = \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} \ \forall\, n \in P$$

$$\therefore f(n-1) = \sum_{k=0}^{\frac{n-3}{2}} \binom{n-k-2}{k} \ \forall\, n \in P$$

We need to show,

$$f(n+1) = \sum_{k=0}^{\frac{n+1}{2}} \binom{n-k}{k} \ \forall\, n \in P$$

Satisfying the condition,

$f(n+1) = f(n) + f(n-1)$

$$\therefore f(n+1) = \sum_{k=0}^{\frac{n+1}{2}} \binom{n-k}{k}$$

$$= \binom{n-0}{0} + \sum_{k=1}^{\frac{n+1}{2}-1} \binom{n-k}{k}$$

$$= 1 + \sum_{k=1}^{\frac{n-1}{2}} \left[\binom{n-k-1}{k} + \binom{n-k-1}{k-1}\right]$$

$$= \binom{n-1}{0} + \sum_{k=1}^{\frac{n-1}{2}} \binom{n-k-1}{k} + \sum_{k=1}^{\frac{n-1}{2}} \binom{n-k-1}{k-1}$$

$$= \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} + \sum_{k=0}^{\frac{n-1}{2}-1} \binom{n-k-2}{k}$$

$$= \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} + \sum_{k=0}^{\frac{n-3}{2}} \binom{n-k-2}{k}$$

$$= f(n) + f(n-1)$$

We have proved it by Mathematical Induction satisfying the condition, $f(n+1) = f(n) + f(n-1)$.

THEOREM – II:

$$F_n = \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} \ \forall\, n \in Q$$

PROOF:

The above theorem can easily be proved by Mathematical Induction.

Let define it as a function,

$$f(n) = \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} \ \forall\, n \in Q$$

We have our set Q as,

$$Q = \{2x \mid x \in \mathbb{Z};\ 2x > 0\}$$

In set Q, it is clear that the elements differ by 2.

By the definition of Fibonacci, we know;

$$f(n+1) = f(n) + f(n-1)$$

BASE PROOF:

$$f(2) = \sum_{k=0}^{0} \binom{2-k-1}{k}$$

$$= \binom{1}{0} = 1$$

$$f(4) = \sum_{k=0}^{1} \binom{4-k-1}{k}$$

$$= \binom{3}{0} + \binom{2}{1} = 1 + 2 = 3$$

We have verified our base step.

INDUCTION HYPOTHESIS:

We have,

$$f(n) = \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} \ \forall\, n \in Q$$

$$\therefore f(n-1) = \sum_{k=0}^{\frac{n-4}{2}} \binom{n-k-2}{k} \ \forall\, n \in Q$$

We need to show,

$$\therefore f(n+1) = \sum_{k=0}^{\frac{n}{2}} \binom{n-k}{k} \ \forall\, n \in Q$$

Satisfying the condition, $f(n+1) = f(n) + f(n-1)$.

$$\therefore f(n+1) = \sum_{k=0}^{\frac{n}{2}} \binom{n-k}{k}$$

$$= \binom{n-0}{0} + \sum_{k=1}^{\frac{n}{2}-1} \binom{n-k}{k}$$

$$= 1 + \sum_{k=1}^{\frac{n-2}{2}} \left[\binom{n-k-1}{k} + \binom{n-k-1}{k-1}\right]$$

$$= \binom{n-1}{0} + \sum_{k=1}^{\frac{n-2}{2}} \binom{n-k-1}{k} + \sum_{k=1}^{\frac{n-2}{2}} \binom{n-k-1}{k-1}$$

$$= \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} + \sum_{k=0}^{\frac{n-2}{2}-1} \binom{n-k-2}{k}$$

$$= \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} + \sum_{k=0}^{\frac{n-4}{2}} \binom{n-k-2}{k}$$

$$= f(n) + f(n-1)$$

We have proved it by Mathematical Induction satisfying the condition, $f(n+1) = f(n) + f(n-1)$.

ALGORITHM - III

This Fi-binomial Conjecture can also be formulated based on the index.

It is known every even and odd number is in the form $2x$ and $2x-1$ respectively.

Now, $x \in \mathbb{N}$;

Considering a Fibonacci number F_n which is of the form $2x-1$.

It can be calculated by Index as,

$$F_{2x-1} = \sum_{k=0}^{n-1} \binom{2x-2-k}{k} \ \forall\, x \in \mathbb{N} \cdots (3)$$

ALGORITHM – IV

Considering a Fibonacci number F_n which is of the form $2x-1$.

It can be calculated by Index as,

$$F_{2x} = \sum_{k=0}^{n-1} \binom{2x-1-k}{k} \ \forall\, x \in \mathbb{N} \cdots (4)$$

Again, Mathematical Induction is enough to prove these formulas.

Up till, now we generalised Fibonacci Number using Binomial Coefficients using two different Formulas.

These two formulae can be summarised into one.

We contend our conjecture,

$$F_n = \begin{cases} \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} \ \forall\, n \in P \\ \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} \ \forall\, n \in Q \end{cases}$$

Where P and Q are given as,

$P = \{2x - 1 \mid x \in \mathbb{Z};\ 2x - 1 > 0\}$

$Q = \{2x \mid x \in \mathbb{Z};\ 2x > 0\}$

ALGORITHM - V

In both the cases the Summation term is same, only the upper limit of the summation differs in the cases.

When n is of the form $2x - 1$, the upper limit of the summation is $\frac{n}{2} - \frac{1}{2}$.

When n is of the form $2x$, the upper limit of the summation is $\frac{n}{2} - 1$.

Actually, for both the cases the upper limit of the summation is the greatest integer function of $\frac{n-1}{2}$.

This can be represented by the floor function as, $\left\lfloor \frac{n-1}{2} \right\rfloor$.

Now, we can summarise both the formulas into one.

We have,

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} \ \forall\, n \in \mathbb{N} \cdots (5)$$

Using Eq. (5), we can find F_n for any value of nth term.

Based on the requirements, one can use any of the formulas for finding the F_nFibonacci Number.

The above equation can easily be proved by Mathematical Induction.

THEOREM – III:

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k}$$

PROOF:

We can prove this by Mathematical Induction.

By the definition of Fibonacci, we know

$$f(n+1) = f(n) + f(n-1)$$

BASE PROOF:

Taking n=1;

$$f(1) = \sum_{k=0}^{0} \binom{1-k-1}{k}$$

$$= \binom{0}{0} = 1$$

For n=2;

$$f(2) = \sum_{k=0}^{0} \binom{2-k-1}{k}$$

$$= \binom{1}{0} = 1$$

We have verified our base step.

INDUCTION HYPOTHESIS:

We have,

$$f(n-1) = \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} \binom{n-k-2}{k}$$

$$f(n) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k}$$

We need to show;

$$f(n+1) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k}$$

$$\therefore f(n+1) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k}$$

$$= \binom{n-0}{0} + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \binom{n-k}{k}$$

Now , we have the upper limit as $\left\lfloor \frac{n}{2} \right\rfloor - 1$.

For a value of x, $\lfloor x \rfloor = 1$.

$\therefore 1 \leq x < 2$ *or* $x \in [1,2)$. *Taking the mean value ofxwe havex* $= \frac{1}{2}$.

$$\therefore \sum_{k=}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k}$$

$$= \binom{n-0}{0} + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \binom{n-k}{k}$$

$$= \binom{n-0}{0} + \sum_{k=1}^{\left(\left\lfloor\frac{n}{2}\right\rfloor - \left\lfloor\frac{1}{2}\right\rfloor\right)} \binom{n-k}{k}$$

$$= 1 + \sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k}{k}$$

$$= 1 + \sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \left[\binom{n-k-1}{k} + \binom{n-k-1}{k-1}\right]$$

$$= 1 + \sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k} + \sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k-1}$$

$$= \binom{n-2}{0} + \sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k} + \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor - 1} \binom{n-k-2}{k}$$

$$= \binom{n-2}{0} + \sum_{k=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k} + \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor - \left\lfloor\frac{1}{2}\right\rfloor} \binom{n-k-2}{k}$$

$$= \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k} + \sum_{k=0}^{\left\lfloor\frac{n-1}{2} - \frac{1}{2}\right\rfloor} \binom{n-k-2}{k}$$

$$= \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k} + \sum_{k=0}^{\left\lfloor\frac{n-2}{2}\right\rfloor} \binom{n-k-2}{k}$$

$$= f(n) + f(n-1)$$

By Mathematical Induction we have proved the value of

$$f(n+1) = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n-k}{k}$$

Satisfying, $f(n+1) = f(n) + f(n-1)$

VI. REPRESENTATION OF FIBONACCI NUMBERS USING FIGURATE NUMBERS

Using the concept of representation of Fibonacci as a sum of Binomial coefficients(Explained in section III.).

That can also be used to represent Fibonacci using Figurate Numbers.

Natural Numbers (linear numbers) can be represented by Binomial Coefficients as,

$$\binom{n}{1} \ \forall\, n \in \mathbb{N}$$

We know that Triangular Numbers which appear on the third diagonal of Pascal's triangle are of the form,

$$T_n = \sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2} \ \forall\, n \in \mathbb{N}$$

Similarly, other 3-D figurate numbers such as Tetrahedral(appear in the fourth diagonal of Pascal's Triangle) and Pentatope Numbers(known as 4-simplex appear in the fifth diagonal of Pascal's Triangle) number are respectively of the form,

$$Te_n = \sum_{m=1}^{n} \sum_{k=1}^{m} \binom{k}{1} = \binom{n+2}{3} \ \forall\, n \in \mathbb{N}$$

$$Ptop_n = \sum_{j=1}^{n} \sum_{m=1}^{j} \sum_{k=1}^{m} \binom{k}{1} = \binom{n+3}{4} \ \forall\, n \in \mathbb{N}$$

Where Te_n and $Ptop_n$ represents the nth Tetrahedral and Pentatope Number respectively.

Similarly, for a r-simplex number which appear on the $(r+1)^{th}$ diagonal of Pascal's Triangle is given as,

$$P_r(n) = \underbrace{\sum_{m=1}^{n} \sum_{i=1}^{m} \cdots \cdots \sum_{k=1}^{j}}_{r-1\ times} \binom{k}{1} = \binom{n+r-1}{r}$$

These Numbers can also be represented as products in the following ways.

$$\therefore \binom{n}{1} = \frac{1}{1!} \prod_{t=0}^{0} (n+t)$$

$$\therefore T_n = \sum_{k=1}^{n} \binom{k}{1} = \frac{1}{2!} \prod_{t=0}^{1} (n+t)$$

$$\therefore Te_n = \sum_{m=1}^{n} \sum_{k=1}^{m} \binom{k}{1} = \frac{1}{3!} \prod_{t=0}^{2} (n+t)$$

$$\therefore P_r(n) = \underbrace{\sum_{m=1}^{n} \sum_{i=1}^{m} \cdots \cdots \sum_{k=1}^{j}}_{r-1\ times} \binom{k}{1} = \frac{1}{r!} \prod_{t=0}^{r-1} (n+t)$$

$$\forall\, r > 0, n \in \mathbb{N}$$

As the r-simplex number appears in the $(r+1)^{th}$ row of the Pascal's Triangle and by recalling the representation of Fibonacci Numbers from Pascal's Triangle (From section. III) it is possible to rewrite the Fibonacci Sequence using Figurate Numbers.

$F(1) = 1$

$F(2) = 1$

$F(3) = 2 = 1 + P_1(1)$

$F(4) = 3 = 1 + P_1(2)$

$F(5) = 5 = 1 + P_1(3) + P_2(1)$

$F(6) = 8 = 1 + P_1(4) + P_2(2)$

$F(7) = 13 = 1 + P_1(5) + P_2(3) + P_3(1)$

$F(8) = 21 = 1 + P_1(6) + P_2(4) + P_3(2)$

By a slight observation one can notice that the general sequence follows as,

$$F(n) = 1 + P_1(n-2) + P_2(n-4) + P_3(n-6) \\ +P_4(n-8) + P_5(n-10) + P_6(n-12) + \ldots$$

Replacing the Figurate Numbers s products as described above, we get

$$F(n) = 1 + \frac{1}{1!}\prod_{t=0}^{0}(n-2+t) + \frac{1}{2!}\prod_{t=0}^{1}(n-4+t) \\ + \frac{1}{3!}\prod_{t=0}^{2}(n-6+t) + \cdots\cdots + \frac{1}{k!}\prod_{t=0}^{k-1}(n-2k+t)$$

This can be represented in a closed form as,

$$F(n) = 1 + \sum_{t=1}^{k}\left\{\frac{1}{t!}\prod_{i=0}^{t-1}(n-2t+i)\right\} \forall n \in \mathbb{N} \cdots (6)$$

Recalling back our sets P and Q,

$P = \{2x - 1 \mid x \in \mathbb{Z};\ 2x - 1 > 0\}$

$Q = \{2x \mid x \in \mathbb{Z};\ 2x > 0\}$

For any $n \in P$, the value of k will run from $k = 1$ to

$$k = \frac{n-1}{2}.$$

Again for any $n \in Q$, the value of k will run from $k = 1$ to

$$k = \frac{n-2}{2}.$$

$$F(n) = \begin{cases} 1 + \sum_{t=1}^{\frac{n-1}{2}}\left\{\frac{1}{t!}\prod_{i=0}^{t-1}(n-2t+i)\right\} \forall n \in P \\ 1 + \sum_{t=1}^{\frac{n-2}{2}}\left\{\frac{1}{t!}\prod_{i=0}^{t-1}(n-2t+i)\right\} \forall n \in Q \end{cases}$$

These two formulas can again be summarised into one.

In both the cases the Summation term is same, only the upper limit of the summation differs in the cases.

When n is of the form $2x - 1$, the upper limit of the summation is $\frac{n}{2} - \frac{1}{2}$.

When n is of the form $2x$, the upper limit of the summation is $\frac{n}{2} - 1$.

Actually, for both the cases the upper limit of the summation is the greatest integer function of $\frac{n-1}{2}$.

This can be represented by the floor function as, $\left\lfloor \frac{n-1}{2} \right\rfloor$.

Now, we can summarise both the formulas into one.

$$\therefore F(n) = 1 + \sum_{t=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\left\{\frac{1}{t!}\prod_{i=0}^{t-1}(n-2t+i)\right\} \forall n \in \mathbb{N}$$

This is a summarised but a much more complicated equation and also not very flexible.

But from this formula it is clear that Fibonacci Sequence can also be represented as the sum of products.

VII. CONCLUSION

We have found a lot of fibonomial identities up to now, which can be used to find the *nth* term of a *Fibonacci* element. Along with the theorems the proofs of the theorem are also provided. We have proved the theorems by the use of Mathematical Induction.

Our *Fibonomial Identity Conjecture* can be summarised as follows.

Consider sets P($P \subset \mathbb{N}$) and Q($Q \subset \mathbb{N}$) such that,

$P = \{2x - 1 \mid x \in \mathbb{Z};\ 2x - 1 > 0\}$

$Q = \{2x \mid x \in \mathbb{Z};\ 2x > 0\}$

An integer $n (n \in \mathbb{N})$, is either of the form $2x$ or $2x - 1$ (as when divisible by 2). So, either $n \in P$ or $n \in Q$.

$$F_n = \begin{Bmatrix} \sum_{k=0}^{\frac{n-1}{2}} \binom{n-k-1}{k} \ \forall\, n \in P \\ \sum_{k=0}^{\frac{n-2}{2}} \binom{n-k-1}{k} \ \forall\, n \in Q \end{Bmatrix}$$

Another important identity using which we can found the *nth*term Fibonacci for any value of n.

$$F_n = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k} \ \forall\, n \in \mathbb{N}$$

We have also shown the representation of Fibonacci sequence using Figurate Numbers which again can be represented by Summation of products. The formula is much more complicated and not so flexible but it will work correctly and yield Fibonacci sequence correctly.

The summarised formula is given below.

$$F(n) = 1 + \sum_{t=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \left\{\frac{1}{t!}\prod_{i=0}^{t-1}(n-2t+i)\right\} \forall\, n \in \mathbb{N}$$

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