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Comparative Analysis of Different Numerical Methods for the Solution of Initial Value Problems in First Order Ordinary Differential Equations

Vibahvari Tukaram Dhokrat

Assistant Professor, K.T.H.M. College, Nashik, Maharashtra, India

ABSTRACT

A mathematical equation which involves a function and its derivatives is called a differential equation. We consider a real-life situation, from this form a mathematical model, solve that model using some mathematical concepts and take interpretation of solution. It is a well-known and popular concept in mathematics because of its massive application in real world problems. Differential equations are one of the most important mathematical tools used in modeling problems in Physics, Biology, Economics, Chemistry, Engineering and medical Sciences. Differential equation can describe many situations viz: exponential growth and de-cay, the population growth of species, the change in investment return over time. We can solve differential equations using classical as well as numerical methods, In this paper we compare numerical methods of solving initial valued first order ordinary differential equations namely Euler method, Improved Euler method, Runge-Kutta method and their accuracy level. We use here Scilab Software to obtain direct solution for these methods.

KEYWORDS: *Differential Equations, Accuracy, local Error, Global Error Step-size*

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INTRODUCTION

Numerical analysis is the continuation of Mathematics. The main aim of numerical analysis is to construct computational methods and provide software for the efficient approximate solution of mathematical problems of all kinds using computers as its main tool. In numerical analysis we design methods that gives accurate approximations which converge to exact solution. Non-linear problems and problems from analysis such as differential equation can be solved by approximate i.e. numerical methods. The techniques for solving differential equation using numerical approximations were developed before computer programming existed. In the Eighteenth century the mathematician Euler (1768),in the Nineteenth century the mathematician Carl (1895) and In the Twentieth century the mathematician Martin Kutta (1905) developed the formulae to describe the solution of initial value problem in first order ordinary differential equations.

For comparative analysis we consider the standard differential equation of first order as follows:

$$y' = f(x, y(x)) \quad \text{where} \quad y(x_0) = y_0.$$

1. Euler Method

This was the oldest and simplest method originated by Leonhard Euler in 1768.It is the first order numerical method for solving ordinary differential equations with given initial conditions. This method is use to analyze a differential equation(D.E.) which uses the idea of linear approximation, where we use small tangent lines over a short distance to approximate the solution to an initial value problem. Consider a D.E.

$$y' = f(x, y(x)) \quad \text{where} \quad y(x_0) = y_0.$$

The formula to solve given D.E. using Euler's method is given by

$y_{n+1} = y_n + h \bullet f(x_n, y_n)$ with $n = 0,1,2,\cdots$ where h is step-size.

Later on, Euler's method is also known as first order Runge-Kutta method.

The local error occurred in this method is proportional to the square of step size and the global error occurred in this method is proportional to step-size.

2. Improved Euler Method

Euler method gives best result when the functions are linear in nature. When functions are non-linear there occurs a truncation error. To remove this drawback Improved Euler's method is introduced. In this method instead of point the arithmetic average of the slopes at x_n and x_{n+1}(that is, at the end points of each subinterval) is used. This method based on two values of dependent variable y_{n+1} and predicted value by Euler method i.e., $\overline{y_{n+1}}$.

$$\overline{y_{n+1}} = y_n + h \bullet f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, \overline{y_{n+1}}\right)\right]$$

This method is also known as Heun's method or second order Runge-Kutta method. The local error occurred in this method is proportional to the cube of step size and the global error occurred in this method is proportional to square of step size.

3. Third order Runge-Kutta method

The third order Runge-Kutta formula is as follows:

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + 4 \bullet k_2 + k_3)$$

$$\text{where} \quad k_1 = h \bullet f(x_n, y_n)$$

$$k_2 = h \bullet f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h \bullet f(x_n + h, y_n - k_1 + 2k_2)$$

This method is also known as Heun's method of order Three.

The local error occurred in this method is proportional to the fourth power of step size and the global error occurred in this method is proportional to cube of step size.

4. Fourth order Runge-Kutta method

It is most popular Runge-Kutta method and widely used in solving the first order D.E. The formula for this method is as follows:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2 \bullet k_2 + 2 \bullet k_3 + k_4)$$

$$\text{where} \quad k_1 = h \bullet f(x_n, y_n)$$

$$k_2 = h \bullet f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h \bullet f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h \bullet f(x_n + h, y_n + k_3)$$

The local error occurred in this method is proportional to the fifth power of step size and the global error occurred in this method is proportional to fourth power of step size.

Application and comparison:

Consider the first order ordinary D.E.as

$$\frac{dy}{dx} = x - y, \quad with \quad y(0) = 1.$$

We first solve it by classical method and then by numerical methods.

1. Classical method:

This is linear D.E.

Comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

Here $P(x) = 1, Q(x) = x$

Integrating factor= $I = e^{\int P(x)dx} = e^{\int 1dx} = e^x$

It's solution is

$$y\ I = \int Q\ Idx + c$$

$$ye^x = xe^x - e^x + c$$

Using initial conditions $y(0) = 1$

$$c = 2$$

The particular solution is

$$y = x - 1 + 2e^{-x}$$

For different values of x we get different solutions for y.

Now we apply four above numerical method to given D.E. one by one. Here we consider the step size 0.2 for each method.

The following table gives the value of y for x from 0 to 2, taking h=0.2.

Sr. No.	X	Exact solution by classical method	Euler's method	Improved Euler method	Third order Runge-Kutta method	Fourth order Runge-Kutta method
1	0	1	1	1	1	1
2	0.2	0.83746	0.80	0.84	0.82368	0.837467
3	0.4	0.74064	0.68	0.7448	0.721139	0.740649
4	0.6	0.69762	0.624	0.702736	0.677912	0.697634
5	0.8	0.69866	0.6192	0.704244	0.68237	0.698669
6	1	0.73576	0.65636	0.74148	0.725165	0.73577
7	1.2	0.80239	0.724288	0.808013	0.798781	0.8024
8	1.4	0.89319	0.8194304	0.898571	0.897175	0.893205
9	1.6	1.00379	0.9355443	1.00883	1.14982	1.0038
10	1.8	1.13060	1.0684355	1.13524	1.14982	1.13061
11	2	1.27067	1.2147484	1.2749	1.29702	1.27068

The above table shows that Euler's method shows more variation with exact solution while Improved method and Fourth order Runge-kutta method gives solution which is closed to exact solution.

Conclusion

The main aim of this paper is to make the comparison between the numerical methods Euler method, Improved Euler method, Third order Runge-kutta method and Fourth order Runge-Kutta method and also with classical solution of first order ordinary D.E. In this paper we showed that the solution obtained by classical method is closed to solution obtained by numerical method. If in numerical method we reduce step size as small as possible to get more accurate solution. Among all these methods Fourth order Runge-kutta method gives more accurate and more efficient solution. Therefore, this method is widely used in solving first order ordinary D.E.

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