International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 5 Issue 5, July-August 2021 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

The Queue Length of a GI/M/1 Queue with Set-Up Period and Bernoulli Working Vacation Interruption

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ABSTRACT

Consider a GI/M/1 queue with set-up period and working vacations. During the working vacation period, customers can be served at a lower rate, if there are customers at a service completion instant, the vacation can be interrupted and the server will come back to a set-up period with probability $p(0 \le p \le 1)$ or continue the working vacation with probability $1-p$, and when the set-up period ends, the server will switch to the normal working level. Using the matrix analytic method, we obtain the steady-state distributions for the queue length at arrival epochs.

KEYWORDS: *GI/M/1; set-up period; working vacation; vacation interruption; Bernoulli*

How to cite this paper: Li Tao "The Queue Length of a GI/M/1 Queue with Set-Up Period and Bernoulli Working Vacation Interruption" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-5 | Issue-5, August 2021, pp.72-76, URL: www.ijtsrd.com/papers/ijtsrd43743.pdf

IJTSRD43743

1. INTRODUCTION

Servi and Finn [1] first introduced the working vacation models and studied an M/M/1 queue, the server commits a lower service rate rather than completely stopping the service during a vacation. Baba [2] considered a GI/M/1 queue with working vacations by the matrix-analytic method. For the vacation interruption models, Li and Tian [3] first introduced and studied an M/M/1 queue with working vacations and vacation interruption. Then, Li et al. [4] analyzed the GI/M/1 queue with working vacations and vacation interruption by the matrix-analytic method. Meanwhile, in some practical situations, it needs some times to switch the lower rate to the normal working level, which we call set-up times. Zhao et al. [5] considered a GI/M/1 queue with set-up period and working vacation and vacation interruption. Bai et al. [6] studied a GI/M/1 queue with set-up period and working vacations.

In this paper, based on the Bernoulli schedule rule we analyze a GI/M/1 queue with set-up period and working vacation and vacation interruption at the same time. Zhang and Shi [7] first studied an M/M/1 queue with vacation and vacation interruption under the Bernoulli rule. In our model, during the working vacation period, if there are customers at a service completion instant, the server can come back to a set-up period with probability $p(0 \le p \le 1)$, not with probability 1, or continue the working vacation with probability $1-p$, which is different from the situation many authors considered before, and when the set-up period ends, the server will switch to the normal busy period. Clearly, the models in [5,6] will be the special cases of the model we consider.

2. Model description and embedded Markov chain

Consider a GI/M/1 queue such that the arrival process is a general distribution process. The server begins a vacation each time when the queue becomes empty and if there are customers arriving in a vacation period, the server continues to work at a lower rate, i.e., the working vacation period is an operation period in lower speed. At a service completion instant, if there are customers in the vacation period, the vacation can be interrupted and the server is resumed to a set-up period with probability $p(0 \le p \le 1)$, or continues the vacation with probability

$\bar{p}$ ($\bar{p}=1-p$), and when the set-up period ends, the server will switch to the normal working level. Otherwise, the server continues the vacation. Meanwhile, if there is no customer when a vacation ends, the server begins another vacation, otherwise, he switches to the set-up period, and after the set-up period, the server switches to the normal busy period.

Suppose τ_n be the arrival epoch of nth customers with $\tau_0=0$. The inter-arrival times $\{\tau_n, n\geq 1\}$ are independent and identically distributed with a general distribution function, denoted by $A(t)$ with a mean $1/\lambda$ and a Laplace Stieltjes transform (LST), denoted by $A^*(s)$. The service times during a normal service period, the service times during a working vacation period, the set-up times and the working vacation times are exponentially distributed with rate μ, η, β and θ, respectively.

Let $L(t)$ be the number of customers in the system at time t and $L_n=L(\tau_n-0)$ be the number of the customers before the nth arrival. Define $J_n=0$, the nth arrival occurs during a working vacation period; $J_n=1$, the nth arrival occurs during a set-up period; $J_n=2$, the nth arrival occurs during a normal service period. Then, the process $\{(L_n,J_n),n\geq 1\}$ is an embedded Markov chain with state space

$\Omega=\{0,0\}\cup\{(k,j),k\geq 1,j=0,1,2\}$.

In order to express the transition matrix of (L_n,J_n), let

$$p_{(i,j),(k,l)}=P(L_{n+1}=k,J_{n+1}=l \mid L_n=i,J_n=j).$$

Meanwhile, we introduce the expressions below

$$a_k=\int_0^\infty \frac{(\mu t)^k}{k!}e^{-\mu t}dA(t),\quad k\geq 0,$$

$$b_k=\int_0^\infty\int_0^t \beta e^{-\beta x}\frac{(\mu(t-x))^k}{k!}e^{-\mu(t-x)}dxdA(t),\quad k\geq 0,$$

$$c_k=\int_0^\infty \bar{p}^k\frac{(\eta t)^k}{k!}e^{-\eta t}e^{-\theta t}dA(t),\quad k\geq 0,$$

$$d_k=\int_0^\infty\sum_{l=0}^{k}\bar{p}^l\int_0^t\frac{(\eta x)^l}{l!}e^{-\eta x}\theta e^{-\theta x}\int_x^t\beta e^{-\beta(y-x)}\times\frac{(\mu(t-y))^{k-l}}{(k-l)!}e^{-\mu(t-y)}dydxdA(t),\quad k\geq 0,$$

$$e_k=\int_0^\infty\sum_{l=1}^{k}\bar{p}^{l-1}p\int_0^t\frac{\eta(\eta x)^{l-1}}{(l-1)!}e^{-\eta x}e^{-\theta x}\int_x^t\beta e^{-\beta(y-x)}\times\frac{(\mu(t-y))^{k-l}}{(k-l)!}e^{-\mu(t-y)}dydxdA(t),\quad k\geq 1,$$

$$f_k=\int_0^\infty \bar{p}^k\int_0^t\frac{(\eta x)^k}{k!}e^{-\eta x}\theta e^{-\theta x}e^{-\beta(t-x)}dxdA(t),\quad k\geq 0,$$

$$g_k=\int_0^\infty \bar{p}^{k-1}p\int_0^t\frac{\eta(\eta x)^{k-1}}{(k-1)!}e^{-\eta x}e^{-\theta x}e^{-\beta(t-x)}dxdA(t),\quad k\geq 1.$$

Using the lexicographic sequence for the states, the transition probability matrix of (L_n,J_n) can be written as the Block-Jacobi matrix

$$\tilde{P}=\begin{pmatrix} B_{00} & A_{01} & & & & \\ B_1 & A_1 & A_0 & & & \\ B_2 & A_2 & A_1 & A_0 & & \\ B_3 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$B_{00}=1-c_0-d_0-f_0;\quad A_{01}=(c_0,f_0,d_0);$

$$A_0=\begin{pmatrix} c_0 & f_0 & d_0 \\ 0 & A^*(\beta) & b_0 \\ 0 & 0 & a_0 \end{pmatrix};\quad A_k=\begin{pmatrix} c_k & f_k+g_k & d_k+e_k \\ 0 & 0 & b_k \\ 0 & 0 & a_k \end{pmatrix},$$

$$B_k=\begin{pmatrix} 1-\sum_{i=1}^{k}(c_i+d_i+e_i+f_i+g_i)-c_0-d_0-f_0 \\ 1-\sum_{i=0}^{k}b_i-A^*(\beta) \\ 1-\sum_{i=0}^{k}a_i \end{pmatrix},\quad k\geq 1.$$

3. Steady-state distribution at arrival epochs

We first define

$$A(z)=\sum_{k=0}^\infty a_kz^k,\ B(z)=\sum_{k=0}^\infty b_kz^k,\ C(z)=\sum_{k=0}^\infty c_kz^k,$$

$$D(z)=\sum_{k=0}^\infty d_kz^k,\ E(z)=\sum_{k=1}^\infty e_kz^k,\ F(z)=\sum_{k=0}^\infty f_kz^k,\ G(z)=\sum_{k=1}^\infty g_kz^k.$$

In this section, we derive the steady-state distribution for (L_n,J_n) at arrival epochs using matrix-geometric approach. In order to derive the steady-state distribution, we need the following three lemmas.

Lemma 3.1.

$$A(z)=A^*(\mu-\mu z),$$

$$B(z)=\frac{\beta[A^*(\mu-\mu z)-A^*(\beta)]}{\beta-\mu(1-z)},$$

$$C(z)=A^*(\theta+\eta-\bar{p}\eta z),$$

$$D(z)=\frac{\theta\beta}{\beta-\mu(1-z)}\frac{[A^*(\theta+\eta-\bar{p}\eta z)-A^*(\beta)]}{\theta+\eta-\bar{p}\eta z-\beta}-\frac{\theta\beta}{\beta-\mu(1-z)}\frac{[A^*(\theta+\eta-\bar{p}\eta z)-A^*(\mu-\mu z)]}{\theta+p\eta z-(\mu-\eta)(1-z)},$$

$$E(z)=\frac{p\eta z\beta}{\beta-\mu(1-z)}\frac{[A^*(\theta+\eta-\bar{p}\eta z)-A^*(\beta)]}{\theta+\eta-\bar{p}\eta z-\beta}-\frac{p\eta z\beta}{\beta-\mu(1-z)}\frac{[A^*(\theta+\eta-\bar{p}\eta z)-A^*(\mu-\mu z)]}{\theta+p\eta z-(\mu-\eta)(1-z)},$$

$$F(z)=\frac{\theta[A^*(\beta)-A^*(\theta+\eta-\bar{p}\eta z)]}{\theta+\eta-\bar{p}\eta z-\beta},$$

$$G(z)=\frac{p\eta z[A^*(\beta)-A^*(\theta+\eta-\bar{p}\eta z)]}{\theta+\eta-\bar{p}\eta z-\beta}.$$

Lemma 3.2. If $\theta>0$, the equation $z=A^*(\theta+\eta-\bar{p}\eta z)$ has a unique root in the range 0<z<1.

Lemma 3.3. If $\theta>0$, $\beta>0$ and $\rho=\lambda/\mu<1$, then the matrix equation $R=\sum_{k=0}^{\infty}R^kA_k$ has the minimal nonnegative solution

$$R=\begin{pmatrix} r_1 & \delta(r_2-r_1) & \Delta\delta(r_3-r_2)-\gamma\delta(r_3-r_1) \\ 0 & r_2 & \Delta(r_3-r_2) \\ 0 & 0 & r_3 \end{pmatrix},$$

where $r_2=A^*(\beta)$, r_1 and r_3 are the unique roots in the range 0<z<1 of equations $z=A^*(\theta+\eta-\bar{p}\eta z)$ and $z=A^*(\mu-\mu z)$, respectively, and $\delta=(\theta+p\eta r_1)/(\theta+\eta-\bar{p}\eta r_1-\beta)$, $\Delta=\beta/[\beta-\mu(1-r_2)]$, $\gamma=\beta/[\theta+p\eta r_1-(\mu-\eta)(1-r_1)]$.

Moreover, we can easily verify that the Markov chain $\tilde{P}$ is positive recurrent if and only if $\theta>0$, $\beta>0$ and $\rho<1$. And the matrix

$$B[R]=\begin{pmatrix} B_{00} & A_{01} \\ \sum_{k=1}^{\infty}R^{k-1}B_k & \sum_{k=1}^{\infty}R^{k-1}A_k \end{pmatrix}$$

$$=\begin{pmatrix} 1-c_0-d_0-f_0 & c_0 & f_0 & d_0 \\ \frac{c_0+f_0}{r_1}-\frac{\delta(r_2-r_1)}{r_1}-\omega & 1-\frac{c_0}{r_1} & \frac{\delta(r_2-r_1)}{r_1}-\frac{f_0}{r_1} & \omega \\ 1-\frac{a_0\Delta(r_3-r_2)}{r_2r_3}+\frac{b_0}{r_2} & 0 & 0 & \frac{a_0\Delta(r_3-r_2)}{r_2r_3}-\frac{b_0}{r_2} \\ \frac{a_0}{r_3} & 0 & 0 & 1-\frac{a_0}{r_3} \end{pmatrix}$$

with $\omega=[\frac{\Delta\delta(r_3-r_2)}{r_2r_3}-\frac{\gamma\delta(r_3-r_1)}{r_1r_3}]a_0+\frac{\delta(r_2-r_1)}{r_1r_2}b_0-\frac{d_0}{r_1}$ has a positive left invariant vector

$$K(1,r_1,\delta(r_2-r_1),\Delta\delta(r_3-r_2)-\gamma\delta(r_3-r_1)) \quad (1)$$

where K ia a random positive real number.

Let (L,J) be the stationary limit of the process (L_n,J_n), and denote

$$\pi_0=\pi_{00};\quad \pi_k=(\pi_{k0},\pi_{k1},\pi_{k2}),\quad k\ge 1,$$

$$\pi_{kj}=P\{L=k,J=j\}=\lim_{n\to\infty}P\{L_n=k,J_n=j\},\quad (k,j)\in\Omega.$$

Theorem 3.4. If $\theta>0$, $\beta>0$ and $\rho<1$, the stationary probability distribution of (L,J) is given by

$$\begin{cases} \pi_{k0}=(1-r_3)\sigma r_1^k, & k\ge 0, \\ \pi_{k1}=(1-r_3)\sigma\delta(r_2^k-r_1^k), & k\ge 0, \\ \pi_{k2}=(1-r_3)\sigma[\Delta\delta(r_3^k-r_2^k)-\gamma\delta(r_3^k-r_1^k)], & k\ge 1, \end{cases}$$

where

$$\sigma=\frac{(1-r_1)(1-r_2)}{(1-r_3)[(1-r_2)+\delta(r_2-r_1)]+\Delta\delta(1-r_1)(r_3-r_2)-\gamma\delta(1-r_2)(r_3-r_1)}.$$

Proof. With the Theorem 1.5.1 in [8], $(\pi_{00},\pi_{10},\pi_{11},\pi_{12})$ is given by the positive left invariant vector Eq. (1), and satisfies the normalizing condition

$$\pi_{00}+(\pi_{10},\pi_{11},\pi_{12})(I-R)^{-1}e=1,$$

where e is a column vector with all elements equal to one. Substituting R into the above relationship, we can get

$$K=\frac{(1-r_1)(1-r_2)(1-r_3)}{(1-r_3)[(1-r_2)+\delta(r_2-r_1)]+\Delta\delta(1-r_1)(r_3-r_2)-\gamma\delta(1-r_2)(r_3-r_1)}=(1-r_3)\sigma.$$

Therefore, we have

$$(\pi_{10},\pi_{11},\pi_{12})=(1-r_3)\sigma(r_1,\delta(r_2-r_1),\Delta\delta(r_3-r_2)-\gamma\delta(r_3-r_1)).$$

Using the Theorem 1.5.1 of Neuts [8], we can obtain

$$\pi_k=(\pi_{k0},\pi_{k1},\pi_{k2})=(\pi_{10},\pi_{11},\pi_{12})R^{k-1},\quad k\ge 1. \quad (2)$$

Taking $(\pi_{10},\pi_{11},\pi_{12})$ and R^{k-1} into Eq. (2), the theorem can be derived.

Then, we discuss the distribution of the queue length L at the arrival epochs. From Theorem 3.4, we have

$$\pi_0=P\{L=0\}=\pi_{00}=(1-r_3)\sigma,$$

$$\pi_k=P\{L=k\}=\pi_{k0}+\pi_{k1}+\pi_{k2}$$
$$=(1-r_3)\sigma[(1-\delta)r_1^k+\delta r_2^k+\Delta\delta(r_3^k-r_2^k)-\gamma\delta(r_3^k-r_1^k)],\quad k\ge 1.$$

The state probability of a server in the steady-state is given by

$$P\{J=0\}=\sum_{k=0}^{\infty}\pi_{k0}=\frac{(1-r_3)\sigma}{1-r_1},$$

$$P\{J=1\}=\sum_{k=1}^{\infty}\pi_{k1}=\frac{\sigma\delta(1-r_3)(r_2-r_1)}{(1-r_1)(1-r_2)},$$

$$P\{J=2\}=\sum_{k=1}^{\infty}\pi_{k2}=\frac{\sigma\Delta\delta(1-r_1)(r_3-r_2)-\sigma\gamma\delta(1-r_2)(r_3-r_1)}{(1-r_1)(1-r_2)}.$$

Theorem 3.5. If $\theta>0$, $\beta>0$ and $\rho<1$, the stationary queue length L can be decomposed as $L=L_0+L_d$, where L_0 is the stationary queue length of a classical GI/M/1 queue without vacation, and follows a geometric distribution with parameter r_3. Additional queue length L_d has a distribution

$$P\{L_d=0\}=\sigma,$$

$$P\{L_d=k\}=\sigma(\delta-1-\gamma\delta)(r_3-r_1)r_1^{k-1}$$
$$+\sigma\delta(\Delta-1)(r_3-r_2)r_2^{k-1},\quad k\ge 1.$$

Proof. The probability generating function of L is as follows:

$$L(z)=\sum_{k=0}^{\infty}\pi_{k0}z^k+\sum_{k=1}^{\infty}\pi_{k1}z^k+\sum_{k=1}^{\infty}\pi_{k2}z^k$$

$$=(1-r_3)\sigma[\frac{1}{1-r_1z}+\delta\frac{(r_2-r_1)z}{(1-r_1z)(1-r_2z)}+\Delta\delta\frac{(r_3-r_2)z}{(1-r_2z)(1-r_3z)}-\gamma\delta\frac{(r_3-r_1)z}{(1-r_1z)(1-r_3z)}]$$

$$=\frac{1-r_3}{1-r_3z}\sigma[\frac{1-r_3z}{1-r_1z}+\delta\frac{(r_2-r_1)(1-r_3z)z}{(1-r_1z)(1-r_2z)}+\Delta\delta\frac{(r_3-r_2)z}{1-r_2z}-\gamma\delta\frac{(r_3-r_1)z}{1-r_1z}]$$

$$=\frac{1-r_3}{1-r_3z}\sigma[1-\frac{(r_3-r_1)z}{1-r_1z}+\delta\frac{(r_3-r_1)z}{1-r_1z}-\delta\frac{(r_3-r_2)z}{1-r_2z}+\Delta\delta\frac{(r_3-r_2)z}{1-r_2z}-\gamma\delta\frac{(r_3-r_1)z}{1-r_1z}]$$

$$=\frac{1-r_3}{1-r_3z}[\sigma+\sigma(\delta-1-\gamma\delta)\frac{(r_3-r_1)z}{1-r_1z}+\sigma\delta(\Delta-1)\frac{(r_3-r_2)z}{1-r_2z}]$$

$$=L_0(z)L_d(z).$$

which completes the proof.

Thus, the mean queue length at the arrival epoch is given by

$$E[L]=\frac{r_3}{1-r_3}+\frac{\sigma(\delta-1-\gamma\delta)(r_3-r_1)}{(1-r_1)^2}+\frac{\sigma\delta(\Delta-1)(r_3-r_2)}{(1-r_2)^2}.$$

4. Steady-state distribution at arbitrary epochs

Now we consider the steady-state distribution for the queue length at arbitrary epochs. And, denote the limiting distribution of $L(t)$: $p_k=\lim_{t\to\infty}P\{L(t)=k\}$.

Theorem 4.1. If $\theta>0$, $\beta>0$ and $\rho<1$, the limiting distribution of $L(t)$ exists. And, we obtain

$$\begin{cases} p_0=1-\sigma\lambda[\frac{\Delta\delta-\gamma\delta}{\mu}+\frac{(1-\Delta)\delta(1-r_3)}{\beta}+\frac{(1-\delta+\gamma\delta)(1-r_3)}{\theta+\eta-\bar{p}\eta r_1}], \\ p_k=(1-r_3)\sigma\lambda[\frac{\Delta\delta-\gamma\delta}{\mu}r_3^{k-1}+\frac{(1-\Delta)\delta(1-r_2)}{\beta}r_2^{k-1}+\frac{(1-\delta+\gamma\delta)(1-r_1)}{\theta+\eta-\bar{p}\eta r_1}r_1^{k-1}], k\ge 1. \end{cases}$$

Proof. From the theory of SMP, the limiting distribution of $L(t)$ has the following expression (see [9]):

$$p_k=\sum_{i=k-1}^{\infty}\pi_{i2}\int_0^{\infty}\frac{(\mu t)^{i+1-k}}{(i+1-k)!}e^{-\mu t}\lambda(1-A(t))dt$$

$$+\sum_{i=k-1}^{\infty}\pi_{i1}\int_0^{\infty}\int_0^t\beta e^{-\beta x}\frac{(\mu(t-x))^{i+1-k}}{(i+1-k)!}e^{-\mu(t-x)}dx\lambda(1-A(t))dt$$

$$+\pi_{k-1,1}\int_0^{\infty}e^{-\beta t}\lambda(1-A(t))dt$$

$$+\sum_{i=k-1}^{\infty}\pi_{i0}\int_0^{\infty}\bar{p}^{i+1-k}\frac{(\eta t)^{i+1-k}}{(i+1-k)!}e^{-\eta t}e^{-\theta t}\lambda(1-A(t))dt$$

$$+\sum_{i=k-1}^{\infty}\pi_{i0}\int_0^{\infty}\sum_{l=0}^{i+1-k}\bar{p}^l\int_0^t\frac{(\eta x)^l}{l!}e^{-\eta x}\theta e^{-\theta x}\int_x^t\beta e^{-\beta(y-x)}$$
$$\times\frac{(\mu(t-y))^{i+1-k-l}}{(i+1-k-l)!}e^{-\mu(t-y)}dydx\lambda(1-A(t))dt$$

$$+\sum_{i=k}^{\infty}\pi_{i0}\int_0^{\infty}\sum_{l=1}^{i+1-k}\bar{p}^{l-1}p\int_0^t\frac{\eta(\eta x)^{l-1}}{(l-1)!}e^{-\eta x}e^{-\theta x}\int_x^t\beta e^{-\beta(y-x)}$$
$$\times\frac{(\mu(t-y))^{i+1-k-l}}{(i+1-k-l)!}e^{-\mu(t-y)}dydx\lambda(1-A(t))dt$$

$$+\sum_{i=k-1}^{\infty}\pi_{i0}\int_0^{\infty}\bar{p}^{i+1-k}\int_0^t\frac{(\eta x)^{i+1-k}}{(i+1-k)!}e^{-\eta x}\theta e^{-\theta x}e^{-\beta(t-x)}dx\lambda(1-A(t))dt$$

$$+\sum_{i=k}^{\infty}\pi_{i0}\int_0^{\infty}\bar{p}^{i-k}p\int_0^t\frac{\eta(\eta x)^{i-k}}{(i-k)!}e^{-\eta x}e^{-\theta x}e^{-\beta(t-x)}dx\lambda(1-A(t))dt$$

$$=b1+b2+b3+b4+b5+b6+b7+b8.$$

We compute each part of the equation and have

$$b1=(1-r_3)\sigma\lambda[\frac{\Delta\delta-\gamma\delta}{\mu}r_3^{k-1}-\Delta\delta\frac{1-A^*(\mu-\mu r_2)}{\mu(1-r_2)}r_2^{k-1}+\gamma\delta\frac{1-A^*(\mu-\mu r_1)}{\mu(1-r_1)}r_1^{k-1}],$$

$$b2=(1-r_3)\sigma\lambda[\Delta\delta\frac{1-A^*(\mu-\mu r_2)}{\mu(1-r_2)}r_2^{k-1}-\Delta\delta\frac{1-r_2}{\beta}r_2^{k-1}-\frac{\beta\delta}{\beta-\mu+\mu r_1}\frac{(1-A^*(\mu-\mu r_1))}{\mu(1-r_1)}r_1^{k-1}+\frac{\delta(1-r_2)}{\beta-\mu+\mu r_1}r_1^{k-1}],$$

$$b3=(1-r_3)\sigma\lambda[\frac{\delta(1-r_2)}{\beta}r_2^{k-1}-\frac{\delta(1-r_2)}{\beta}r_1^{k-1}],$$

$$b4=(1-r_3)\sigma\lambda\frac{1-r_1}{\theta+\eta-\bar{p}\eta r_1}r_1^{k-1},$$

$$b5=(1-r_3)\sigma\frac{(1-r_3)\sigma\lambda}{\beta-\mu+\mu r_1}\frac{\theta\beta r_1^{k-1}}{\theta+\eta-\bar{p}\eta r_1-\beta}(\frac{1-r_1}{\theta+\eta-\bar{p}\eta r_1}-\frac{1-r_2}{\beta})-\frac{(1-r_3)\sigma\lambda\theta\gamma r_1^{k-1}}{\beta-\mu+\mu r_1}[\frac{1-r_1}{\theta+\eta-\bar{p}\eta r_1}-\frac{1-A^*(\mu-\mu r_1)}{\mu(1-r_1)}],$$

$$b6=\frac{(1-r_3)\sigma\lambda}{\beta-\mu+\mu r_1}\frac{p\eta\beta r_1^{k}}{\theta+\eta-\bar{p}\eta r_1-\beta}(\frac{1-r_1}{\theta+\eta-\bar{p}\eta r_1}-\frac{1-r_2}{\beta})-\frac{(1-r_3)\sigma\lambda p\eta\gamma r_1^{k}}{\beta-\mu+\mu r_1}[\frac{1-r_1}{\theta+\eta-\bar{p}\eta r_1}-\frac{1-A^*(\mu-\mu r_1)}{\mu(1-r_1)}],$$

$$b7=\frac{(1-r_3)\sigma\lambda\theta r_1^{k-1}}{\theta+\eta-\bar{p}\eta r_1-\beta}(\frac{1-r_2}{\beta}-\frac{1-r_1}{\theta+\eta-\bar{p}\eta r_1}),$$

Then, using these expressions, the theorem can be obtained by some computation.

Let $\tilde{L}$ denote the steady-state system size at an arbitrary epoch, the mean of $\tilde{L}$ can be given by

$$E[\tilde{L}]=\sum_{k=0}^{\infty}kp_k=(1-r_3)\sigma\lambda[\frac{\Delta\delta-\gamma\delta}{\mu(1-r_3)^2}+\frac{(1-\Delta)\delta}{\beta(1-r_2)}+\frac{1-\delta+\gamma\delta}{(1-r_1)(\theta+\eta-\bar{p}\eta r_1)}].$$

Remark 4.2. If $p=1$, the system reduces to the model described in [4], and if $p=0$, the system becomes a GI/M/1 queue with set-up period and multiple working vacations [6].

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