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A Complete Ideals Derivation in Bcik – Algebras

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ABSTRACT

In this paper, introduced a new notions of N-sub algebras. (closed, commutative, retrenched) N-ideals, $\theta -$negative functions, and ∝-translations are introduced, and related properties are investigated. Characterizations of an N-sub algebra and a (Commutative) N-ideal are given. Relations between an N-sub algebra an N-ideal and commutative N-ideal are discussed. We verify that every ∝-translations of an N-sub algebra (resp. N-ideal) is a retrenched N-sub algebra (resp.retrenched N-ideals).

Already introduced a notationof prime ideal in BCIK – algebra show that it is equivalent to the last definition of prime ideal in lower BCIK – Semi lattices. Then we attempt to generalized some useful theorems about prime ideals, in BCIK-algebras, instead of lower BCIK-Semi lattices. We already investigate some results for PI-lattices being a new classical of BCIK– lattices. Specially, we prove that any Boolean lattice is a PI-lattice and we show that if X is a PI-lattice with bounded commutative, then X is an involutory BCIK-algebra if and only if X is a commutative BCIK-algebra.

***KEYWORDS**: BCIK-algebra, N-sub algebra (closed, commutative, retrenched) N-ideal, θ- negative functions, ∝-translations, prime ideals, maximal ideals.*

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1. INTRODUCTION

In 2021, S Rethina Kumar [1] defined BCK – algebra in this notion originated fromtwo different sources: (Combination of two algebraic notions BCK and BCI) one of them is based on the set theory the other is form the classical and non – classical propositional calculi. They are two important classes of logical algebras, and have applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Also S Rethina Kumar introduced the notion of BCIK-algebra which is a generalization of a BCIK-algebra of a BCIK-algebra [1]. Several properties on BCIK-algebra are investigated in the papers [1,5].But differently deal no negative meaning of information is suggested, now feel a need to deal with negative information. To do so, also feel a need to supply mathematical tool. To attain such object, introduced and use new function which is called negative valued function. The important achievement of this article is that one can deal with positive and negative information simultaneously by combining ideals in this article and already well known positive information.

In this paper, discuss the ideal theory BCIK-algebra based on negative-valued functions. Introduced the notions of N-sub algebras, (closed, commutative, retrenched) N-ideals. $\theta -$ negative functions and ∝-translations, and then investigate several properties. Give characterizations of an N-sub algebra and a (commutative) N-ideal. Discuss relations between an N-sub algebra, an N-ideal and commutative N-ideal. Show that every ∝-translations of an N-sub algebra (resp. N-ideal) is a retrenched N-sub algebra (resp. retrenched N-ideal).

2. [1,2] Preliminaries

Definition 2.1. BCIK algebra

Let X be a non-empty set with a binary operation * and a constant 0. Then (X,*,0) is called a BCIK Algebra, if it satisfies the following axioms for all x,y,z Є X:

(BCIK-1) x*y = 0, y*x = 0, z*x = 0 this imply that x = y = z.

(BCIK-2) ((x*y)*(y*z))*(z*x) = 0.

(BCIK-3) (x*(x*y))*y = 0.

(BCIK-4) x*x = 0, y*y = 0, z*z = 0.

(BCIK-5) 0*x =0, 0*y = 0, 0*z = 0.

For all x,y,z Є X. An inequality ≤ is a partially ordered set on X can be defined x ≤ y if and only if

(x*y)*(y*z) = 0.

Properties 2.2.I any BCIK – Algebra X, the following properties hold for all x,y,z Є X:

1. 0 Є X.
2. x*0 = x.
3. x*0 = 0 implies x = 0.
4. 0*(x*y) = (0*x)*(0*y).
5. X*y = 0 implies x = y.
6. X*(0*y) = y*(0*x).
7. 0*(0*x) = x.
8. x*y Є X and x Є X imply y Є X.
9. (x*y)*z = (x*z)*y
10. x*(x*(x*y)) = x*y.

11. (x*y)*(y*z) = x*y.
12. 0 ≤ x ≤ y for all x,y Є X.
13. x ≤ y implies x*z ≤ y*z and z*y ≤ z*x.
14. x*y ≤ x.
15. x*y ≤ z ⇔ x*z ≤ y for all x,y,z Є X
16. x*a = x*b implies a = b where a and b are any natural numbers (i.e)., a,b Є N
17. a*x = b*x implies a = b.
18. a*(a*x) = x.

Definition 2.3.[1]Let X be a BCIK – algebra. Then, for all x, y, z Є X:
1. X is called a positive implicative BCIK – algebra if (x*y)*z = (x*z)*(y*z).
2. X is called an implicative BCIK – algebra if x*(y*x) = x.
3. X is called a commutative BCIK – algebra if x*(x*y) = y*(y*x).
4. X is called bounded BCIK – algebra, if there exists the greatest element 1 of X, and for any xЄ X, 1*x is denoted by GG_x,
5. X is called involutory BCIK – algebra, if for all x Є X, GG_x = x.

Definition 2.4.Let X be a bounded BCIK-algebra. Then for all x, y Є X:
1. G1 = 0 and G0 = 1,
2. GG_x≤ x that $GG_x = G(G_x)$,
3. $G_x * G_y$≤ y*x,
4. y ≤ x implies $G_x \le G_y$,
5. $G_{x*y} = G_{y*x}$
6. $GGG_x = G_x$.

Theorem 2.5.Let X be a bounded BCIK-algebra. Then for any x, y Є X, the following hold:
1. X is involutory,
2. x*y = $G_y * G_x$,
3. x*G_y = y * G_x,
4. x ≤ G_y implies y ≤ G_x.

Theorem 2.6. Every implicative BCIK-algebra is a commutative and positive implicative BCIK-algebra.

Definition 2.7.Let Xbe a BCIK-algebra. Then:
1. X is said to have bounded commutative, if for any x, y Є X, the set A(x,y) = {t Є X : t*x ≤ y} has the greatest element which is denoted by x o y,
2. (X,*,≤) is called a BCIK-lattices, if (X,≤) is a lattice, where ≤ is the partial BCIK-order on X, which has been introduced in Definition 2.1.

Definition 2.8.Let X be a BCIK-algebra with bounded commutative. Then for all x, y, z Є X:
1. y ≤ x o (y*x),
2. (x o z)*(y o z) ≤ x*y,
3. (x*y)*z = x*(y o z),
4. If x ≤ y, then x o z ≤ y o z,
5. z*x ≤ y ⇔ z ≤ x o y.

Theorem 2.9.Let X be a BCIK-algebra with condition bounded commutative. Then, for all x, y, z Є X, the following are equivalent:
1. X is a positive implicative,
2. x ≤ y implies x o y = y,
3. x o x = x,
4. (x o y) * z = (x*z) o (y*z),
5. x o y = x o (y*x).

Theorem 2.10. Let X be a BCIK-algebra.
1. If X is a finite positive implicative BCIK-algebra with bounded and commutative the (X,≤) is a distributive lattice,
2. If X is a BCIK-algebra with bounded and commutative, then X is positive implicative if and only if (X,≤) is an upper semilattice with x ∨ y = x o y, for any x, y Є X,
3. If X is bounded commutative BCIK-algebra, then BCIK-lattice (X,≤) is a distributive lattice, where x ∧ y = y*(y*x) and x ∨ y= $G(G_x \wedge G_y)$.

Theorem 2.11. Let X be an involutory BCIK-algebra, Then the following are equivalent:
1. (X,≤) isa lower semilattice,
2. (X,≤) is an upper semi lattice,
3. (X,≤) is a lattice.

Theorem 2.12. Let X be a bounded BCIK-algebra. Then:
1. every commutative BCIK-algebra is an involutory BCIK-algebra.
2. Any implicative BCIK-algebra is a Boolean lattice (a complemented distributive lattice).

Theorem 2.13.Let X be a BCK-algebra, Then, for all x, y, z Є X, the following are equivalent:
1. X is commutative,
2. x*y = x*(y*(y*x)),
3. x*(x*y) = y*(y*(x*(x*y))),
4. x≤ y implics x = y*(y*x).

Definition 2.14. Let I be a nonempty subset of BCIK-algebra X containing 0. I is called an ideal of X if $y * x \in I \text{ and } x \in I \text{ imply } y \in I, for\ any\ x, y \in X$. Clearly {0} is an ideal of X and we write 0 is an ideal of X, for convenience. An ideal I is called proper, if $I \neq X$ and is called closed, if $if\ x * y \in I, for\ all\ x, y \in I$. The BCIK-part of X is a closed ideal of X. Let S be a nonempty subset of X. We call the least ideal of X containing S, the generated ideals of X by S and is denoted by (S).x

If A and B are sub algebras of X, then we usually denote A+B for $(A \cup B)$. Moreover. A+B is a closed ideal of X. If X is a BCIK-algebra, then BCIK-part of X is a closed ideal of X and p- Semi simple part of X is a sub algebra of X. If X is a lower BCIK-Semi lattices, then for any $x, y \in X$, we have $(x) \cap (y) = (x \wedge y)$. Let A be an ideal of a BCIK-algebra X. Then the relation 0 defined by $(x, y) \in 0 \Leftrightarrow x * y, y * x \in A$ is a congruence relation on X. We usually denote A_x for $[x] = \{y \in X | (x, y) \in \theta\}$. Moreover, A_0 is a closed ideal of ideal of BCIK-algebra X. In fact, it is the greatest closed ideal contained in A. Assume that $\frac{X}{A} = \{A_x | x \in X\}$. Then

$(X/A,*,A_o)$ is a BCIK-algebra , where $A_x * A_y = A_{x*y}$. For all $x, y \in X$.

Let X and Y be two BCIK-algebras. A map $f: X \to Y$ is called a BCIK-homomorphism if $f(x*y) = f(x) * f(y)$ for all $x, y \in X$. If $f: X \to Y$ is a BCIK-homomorphism, then the set $\ker(f) = f^{-1}(0)$ is a closed ideal of X. A homomorphism is one to one if and only if $\ker(f) = \{0\}$. The homomorphism f is called epimorphism if f is onto. Moreover, an isomorphism is a homomorphism. Which is both one to one and onto. Note that, if $f: X \to Y$ is a BCIK-homomorphism, then f(0)-0. An element x of BCIK-algebra is called nilpotent if $0 * x^n$-0, for some $n \in N$. A BCIK-algebra is called nilpotent if any element of X is nilpotent.

Theorem 2.15. BCIK-algebra X is nilpotent if and only if every ideal of X is closed.

Theorem 2.16. Let S be a nonempty subset of a BCIK-algebra X and $A = \{x \in X | (\ldots.((x * a_1) * \ldots.) * a_n = 0$, for some $n \in N$ and some $a_1, \ldots, a_n \in S\}$. Then $(S) = A \cup \{0\}$. Especially, if S contains a nilpotent of X, then (S)=A, Moreover, if I is an ideal of X,

then $(A \cup S) = \{x \in X | (\ldots.((x * a_1) * a_2) * \ldots.) a_n \in A\}$, for some $n \in N \text{ and } a_1, \ldots, a_n \in S\}$.

Definition 2.17. A proper ideal I of BCIK-algebra X is called an irreducible ideal if $A \cap B = I$ implies $A = I \text{ or } B = I$, for any ideal A and B of X.

Definition 2.18. Let X be a BCIK-algebra. A proper ideal M of X is called a maximal ideal if $(M \cup \{x\}) = X \text{ for any } x \subset X \backslash M$, where $(M \cup \{x\})$ is an ideal generated by $M \cup \{x\}$. Note, M is a maximal ideal of X if and only if $M \subseteq A \subseteq X$ *implies that* $M = A \text{ or } A = X, \text{for any ideal } A \text{ of } X$.

Theorem 2.19. Let X and Y be two BCIK-algebra and $f: X \dashrightarrow Y$ be a BCIK-algebra epimorphism. If $A = \ker(f)$, then $\alpha: X/A \neg Y$ which defined by $a(A_x) = f(x)$ is a BCIK-isomorphism.

Lemma 2.20. Let I and J be two ideals of BCIK-algebra X such $I \subseteq J$. Denote J/I-$\{I_x \in X \backslash x \in J\}$. Then

1. $x \in J$ if and only if $I_x \in J/$I, for any $x \in X$.
2. J/I=$\{I_x \in X/I \,| x \in J\}$ is an ideal of X/I.
3. *Let* I be a closed ideal of X. If S and T are sets of all ideals of X and X/I, respectively, then the map $g: S \longrightarrow T$ defined by $g(J) = J/I$, is a bijective map. The inverse of g is the map $f: T \to S$, is defined by $f(J) = \cup \{I_x | I_x \in J\}$.

Definition 2.21. A proper ideal I of lower BCIK-semi lattice X is called prime if $x \wedge y \in I \text{ implies } x \in I \text{ or } y \in I$. Let $\{X_i\}_{i \in I}$ be a family of BCIK-algebra. Then $\prod_{i \in I} X_i$ is a BCIK-algebra and the map $\pi_j: \prod_{i \in I} X_i \to X_i$ is the i-th natural projection map.

Definition 2.22. A BCIK-algebra X is a subdirect product of BCIK-algebra family $\{X_i\}_{i \in I}$ if there is an one to one BCIK-homomorphism$f: X \to \prod_{i \in I} X_i$such that $\pi_i(f(X)) = X_i$, where $\pi_j: \prod_{i \in I} X_i \to X_i$ is the i-th natural projection maps for all $i \in I$. Moreover, the map f is called sub direct embedding.

3. N-sub algebras and (commutative) N-ideals

Denote by $F(X, [-1,0])$ the collection of functions from a set X to [-1,0], say that an element of $F(X, [-1,0])$ is a negative-valued function from X to [-1,0] (briefly, N-function on X) By an N-structure we mean an ordered pair (X, φ) of X and an N-function φ on X. In what follows, let X denote a BCIK-algebra and φ an N-function on X unless otherwise specified.

Definition 3.1 By a sub algebra of X based on N-function φ (briefly, N-sub algebra of X), we mean an N-structure (X, φ) in which φ satisfies the following assertion: $(\forall\, x, y \in X)(\varphi(x*y) \le \max\{\varphi(x), \varphi(y)\})$. For any N-function φ on X and $t \in [-1,0)$, the set $C(\varphi: t) := \{x \in X | \varphi(x) \le t\}$ is called a closed $(\varphi$,t)-cut of φ, and the set $O(\varphi: t) := \{x \in X | \varphi(x) < t\}$ is called an open (φ, t)-cut of φ.

Theorem 3.2. Let (X, φ) be an N-structure of X and φ. Then (X, φ) is an N-sub algebra of X if and only if every non-empty closed $(\varphi$,t)-cut of φ is a sub algebra of X for all $t \in [-1,0)$.

Proof. Assume that (X, φ) is an N-sub algebra of X and let $t \in [-1,0)$ be such that $C(\varphi; t) \neq$. Let $x, y \in C(\varphi; t)$. Then $\varphi(x) \le t$ and $\varphi(y) \le t$. It follows that $\varphi(x, y) \le \max\{\varphi(x), \varphi(y)\} \le t$ so that $x * y \in C(\varphi; t)$. Hence $C(\varphi; t)$ is a sub algebra of X.

Conversely, suppose that every non-empty closed (φ, t)-cut of X is a sub algebra of X for all $t \in [-1,0)$. If (X, φ) is not an N-sub algebra of X, then $\varphi(a*b) > t_0 \ge \max\{\varphi(a), \varphi(b)\}$ for some $a, b \in X$ and $t \in [-1,0)$. Hence $a, b \in C(\varphi; t_0)$ and $a, b \notin C(\varphi; t_0)$. This a contradiction. Thus $\varphi(x*y) \le \max\{\varphi(x), \varphi(y)\}$ for all $x, y \in X$.

Corollary 3.3. If (X, φ) is an N-sub algebra of X, then every non-empty open $(\varphi, t) -$ cut of X is a sub algebra of X for all $t \in [-1,0)$.

Proof. Straightforward.

Lemma 3.4. Every N-sub algebra (X, φ) of X satisfies the following inequality: $(\forall x \in X)(\varphi(x) \ge \varphi(\theta))$.

Proof. Note that $x * x = \theta$ for all $x \in X$. We have $\varphi(\theta) = \varphi(x*x) \le \max\{\varphi(x) \cdot \varphi(x)\} = \varphi(x)$ for all $x \in X$.

Proposition 3.5. If every N-sub algebra (X, φ) of X satisfies the following inequality: $(\forall x, y \in X)(\varphi(x*y) \le \varphi(y))$, then φ is a constant function.

Proof. Let $x \in X$, have $\varphi(x) = \varphi(x*\theta) \le \varphi(\theta)$. It follows from that $\varphi(x) = \varphi(\theta)$, and so φ is a constant function.

Definition 3.6. By an ideal of X based on N-function φ (briefly, N-ideal of X), we mean an N-structure (X, φ) in which φ satisfies the following assertion: $(\forall x, y \in X)(\varphi(0) \le \varphi(x) \le \max\{\varphi(x*y), \varphi(y)\})$.

Example 3.7. Let X=$\{\theta, a, b, c\}$ be a set with the following Cayley table:

*	θ	A	B	c
θ	θ	θ	θ	θ
A	A	θ	θ	a
B	B	A	θ	b
C	C	C	C	θ

Then $(X,*,0)$ is a BCIK-algebra. Define an N-function φ by

X	θ	A	b	c
φ	-0.7	-0.5	-0.5	-0.3

It is easily verify that (X, φ) is both an N-sub algebra and an N-ideal of X.

Example 3.8. Consider a BCIK-algebra $X \coloneqq Y \times Z$, where $(Y,*,\theta)$ is a BCIK-algebra and $(Z,-,0)$ is the adjoint BCIK-algebra of the additive group $(Z,+,0)$ of integers. Let φ be an N-function on X defined by

$\varphi(x) = \begin{cases} t \; if \; x \in Y \times (N \cup \{0\}), \\ \quad 0 \; otherwise \end{cases}$ for all $x \in X$ where N is the set of all natural numbers and t id fixed in [-1,0), and so (X,φ) is an N-ideal of X.

Proposition 3.9. If (X,φ) is an N-ideal of X, then $(\forall x,y \in X)(x \preccurlyeq y \Longrightarrow \varphi(x) \le \varphi(y))$.

Proof. Let $x,y \in X$ be such that $x \preccurlyeq y$. Then $x * y = \theta$, and so $\varphi(x) \le \max\{\varphi(x*y),\varphi(y)\} = \max\{\varphi(\theta),\varphi(y)\} = \varphi(y)$. This completes the proof.

Proposition 3.10. Let (X,φ) be an N-ideal of X. Then the following are equivalent:

1. $(\forall x,y \in X)\Big(\varphi(x*y) \le \varphi\big((x*y)*y\big)\Big)$,
2. $(\forall x,y,z \in X)\Big(\varphi\big((x*z)*(y*z)\big) \le \varphi\big((x*y)*z\big)\Big)$.

Proof. Assume that (i) is valid and let $x,y,z \in X$. Since $\Big(\big(x*(y*z)\big)*z\Big)*z = \big((x*z)*(y*z)\big)*z \preccurlyeq (x*y)*z$, it follows from that $\varphi(((x*(y**z))*z)*z) \le \varphi((x*y)*z)$, we have $\varphi\big((x*z)*(y*z)\big) = \varphi((x*(y*z))*z) \le \varphi(((x*(y*z))*z)*z) \le \varphi((x*y)*z)$. Conversely suppose that (ii) holds. If we use Z instead of y in (ii), then $\varphi(x*z) = \varphi\big((x*z)*\theta\big) = \varphi((x*z)*(z*z)) \le \varphi((x*z)*z)$ for all $x,z \in X$. This proves (i)

Theorem 3.11. For any sub algebra (res. Ideal) U of X, there exists an N-function φ such that (X,φ) is an N- sub algebra (resp. N-ideal) of X and $C(\varphi,t) = U$ for some $t \in [-1,0)$.

Proof. Let U be a sub algebra (resp. ideal) of X and let φ be an N-function on X defined by

$\varphi(x) = \begin{cases} 0 \; if \; x \notin U \\ t \; if \; x \in U \end{cases}$ where t is fixed in [-1,0). Then (X,φ) is an N-sub algebra (resp. N-ideal) of X and $C(\varphi;t) = U$.

Theorem 3.12. Let (X,φ) be an N-structure of X and φ. Then (X,φ) is an N-ideal of X if and only if it satisfies: $(\forall t \in [-1,0))(C(\varphi;t) \ne \quad \Rightarrow C(\varphi;t)$ is an ideal of X).

Proof. Assume that (X,φ) is an N-ideal of X. Let $t \in [-1,0)$ be such that $C(\varphi;t) \ne$. Obviously, $\theta \in (\varphi;t)$. Let $x,y \in X$ be such that $x*y \in C(\varphi;t)$ and $y \in C(\varphi;t)$. Then $\varphi(x*y) \le t \; and \; \varphi(y) \le t$. It follows from that $\varphi(x) \le \max\{\varphi(x*y),\varphi(y)\} \le t$, so that $x \in C(\varphi;t)$. Hence $C(\varphi;t)$ is an ideal of X.

Conversely, suppose that is valid. If there exists $a \in X$ such that $\varphi(\theta) > \varphi(a), then \; \varphi(\theta) > t_a \ge \varphi(a)$ for some $t_a \in [-1,0)$.x Then $\theta \notin C(\varphi;t_a) which \; is \; a$ contradiction. Hence $\varphi(\theta) \le \varphi(x)$ for all $x \in X$. Now, assume that there exists $a,b \in X$ such that $\varphi(a) > \max\{\varphi(a*b),\varphi(b)\}$. Then there exists $s \in [-1,0)$ such that $\varphi(a) > \max\{\varphi(a*b),\varphi(b)\}$. It follows that $a*b \in C(\varphi;s) \; and \; b \in C(\varphi;s)$, but $a \notin C(\varphi;s)$. This is impossible, and so $\varphi(x) \le \max\big\{\varphi(x*y),\varphi(y)\big)$ for all $x,y \in X$. Therefore (X,φ) is an N-ideal of X.

Corollary 3.13. If (X,φ) is an N-ideal of X, then every non-empty open (φ,t) −cut of X is an ideal of X for all $t \in [-1,0)$.

Proof. Straightforward.

Proposition 3.14. Let (X,φ) is an N-ideal of X. If X satisfies the following assertion: $(\forall x,y,z \in X)(x*y \preccurlyeq z)$then $\varphi(x) \le \max\{\varphi(y),\varphi(z)\}$ for all $x,y,z \in X$.

Proof. Assume that $(\forall x,y,z \in X)(x*y \preccurlyeq z)$ is valid in X. Then $\varphi(x*y) \le \max\big\{\varphi\big((x*y)*z\big),\varphi(z)\big\} = \max\{\varphi(\theta),\varphi(z)\} = \varphi(z)$ for all $x,y,z \in X$. It follows that $\varphi(x) \le \max\{\varphi(x*y),\varphi(y)\} \le \max\{\varphi(y),\varphi(z)\}$ for all $x,y,z \in X$. This completes the proof.

Theorem 3.15. For any BCIK-algebra X, every N-ideal is an N-sub algebra.

Proof. Let (X,φ) be an N-ideal of a BCIK-algebra X and let $x,y \in X$. Then $\varphi(x*y) \le \max\{\varphi\big((x*y)*x\big)\varphi(x)\} = \max\{\varphi\big((x*x)*y\big),\varphi(x)\} = \max\{\varphi(\theta*y),\varphi(x)\} = \max\{\varphi(\theta),\varphi(x)\} \le \max\{\varphi(x),\varphi(y)\}$. Therefore (X,φ) is an N-sub algebra of X.

The converse of Theorem 3.15. may not be true in general as seen in the following example.

Example 3.16. Consider a BCIK-algebra $X = \{\theta,1,2,3,4\}$ with the following Cayley table:

$*$	θ	1	2	3	4
θ	θ	θ	θ	θ	θ
1	1	θ	θ	θ	θ
2	2	1	θ	1	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Define an N-function φ on X by

X	0	1	2	3	4
φ	-0.8	-0.8	-0.2	-0.7	-0.4

Then (X,φ) is an N-sub algebra of X. But it is not an N-ideal of X since $\varphi(2) = -0.2 > -0.7 = \max\{\varphi(2*3),\varphi(3)\}$.

The following example show that Theorem 3.15. is not valid in a BCIK-algebra X, that is, if X is a BCIK-algebra then an N-ideal (X,φ) may not be an N-sub algebra for some N-function φ on X.

Example 3.17. Consider the N-ideal. Take $x = (\theta,0) and \; y = (\theta,1)$. Then $z \coloneqq x*y = (\theta,0)*(0,1) = (0,-1)$ and so $\varphi(x*y) = \varphi(z) = 0 > t = \max\{\varphi(x),\varphi(y)\}$. Therefore (X,φ) is not an N-sub algebra of X. For any element w of X, we consider the set $X_w \coloneqq \{x \in X | \varphi(x) \le \varphi(w)\}$. Obviously, $w \in X_w, and \; so \; X_w$ is a non-empty subset of X.

Theorem 3.18. Let w be an element of X. If (X,φ) is an N-ideal of X, then the set X_w is an ideal of X.

Proof. Obviously, $\theta \in X_w$. Let $x,y \in X$ be such that $x*y \in X_w \; and \; y \in X_w$. Then $\varphi(x*y) \le \varphi(w) and \; \varphi(y) \le \varphi(w)$. Since (X,φ) is an N-ideal of X, it follows that $\varphi(x) \le \max\{\varphi(x*y),\varphi(y)\} \le \varphi(w)$ so that $x \in X_w$. Hence X_w is an ideal of X.

Theorem 3.19. Let w be an element of X and let (X,φ) be an N-structure of X and φ. Then

1. If X_w is an ideal of X, then (X,φ) satisfies the following assertion: $(\forall x,y,z \in X)(\varphi(x) \ge \max\{\varphi(y*z),\varphi(z)\} \Longrightarrow \varphi(x) \ge \varphi(y))$.

2. If (X,φ) satisfies $\varphi(0) \leq \varphi(x)$ for all $x \in X$, then X_w is an ideal of X.

Proof. (i) Assume that $X_w$is an ideal of X for each $w \in X$. Let $x,y,z \in X$ be such that $\varphi(x) \geq \max\{\varphi(y*z),\varphi(z)\}$. Then $y*z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$, that is, $\varphi(y) \leq \varphi(x)$.

(ii)Suppose that (X,φ) satisfies $\varphi(\theta) \leq \varphi(x)$ for all $x \in X$. For each $w \in X$, let $x,y \in X$ be such that $x,y \in X_w$ and $y \in X_w$. Then $\varphi(x*y) \leq \varphi(w)$and $\varphi(y) \leq \varphi(w)$, which imply that $\max\{\varphi(x*y),\varphi(y)\} \leq \varphi(w)$. We have $\varphi(w) \geq \varphi(x)$ and so $x \in X_w$. Obviously $\theta \in X_w$. Therefore X_w is an ideal of X.

Definition 3.20. Let X be a BCIK-algebra. An N-ideal (X,φ) is said to be closed if it also an N-sub algebra of X.

Example 3.21. Let $X = \{\theta,1,a,b,c\}$ be a BCIK-algebra with the following Cayley table:

$*$	θ	1	A	b	c
θ	θ	θ	A	b	c
1	1	θ	a	b	c
a	a	A	θ	c	b
b	b	B	C	θ	a
c	c	C	B	a	θ

Let φ be an N-function on X define by

X	θ	1	a	b	c
φ	-0.9	-0.7	-0.6	-0.2	-0.2

Then (X,φ) is a closed N-ideal of X.

Theorem 3.22. Let X be a BCIK-algebra and let φ be defined by

$\varphi(x) = \begin{cases} t_1 \; if \; x \in X_+, \\ t_2 \; otherwise \end{cases}$ where $t_1,t_2 \in [-1,0)$ with $t_1 < t_2$ and $X_1 = \{x \in X|\theta \preccurlyeq x\}$. Then (X,φ) is a closed N-ideal of X.

Proof. Since $\theta \in X_+$, we have $\varphi(\theta) = t_1 \leq \varphi(x)$ for all $x \in X$. Let$x,y \in X$. If $x \in X_+$, then $\varphi(x) = t_1 \leq \max\{(\varphi(x*y),\varphi(y)\}$. Assume that $x \notin X_+$. If $x*y \in X_+$ then $y \notin X_+$; and if $y \notin X_+$ then $x*y \notin X_+$. In either case, we get $\varphi(x) = t_2 = \max\{\varphi(x*y),\varphi(y)\}$. For every $x,y \in X$, if any one of x and y does not belong to X_+, then $\varphi(x*y) \leq t_2 = \max\{\varphi(x),\varphi(y)\}$. If $x,y \in X_+$, then $x*y \in X_+$ and so $\varphi(x*y) = t_1 = \max\{\varphi(x),\varphi(y)\}$. Therefore (X,φ) is a closed N-ideal of X.

Definition 3.23. Let be a BCIK-algebra. If an N-function φ on X satisfies the following condition: $(\forall x \in X)(\varphi(\theta * x0 \leq \varphi(x))$, then we say that φ-negative function.

Proposition 3.24. Let X be a BCIK-algebra. If (X,φ) is a closed N-ideal of X, then φ is a θ −negative function.

Proof. For any $x \in X$, we have $\varphi(\theta*x) \leq \max\{\varphi(\theta),\varphi(x)\} \leq \max\{\varphi(x),\varphi(x)\} = \varphi(x)$.

Therefore φ is a θ −negative function.

We provide a condition for an N-ideal to be closed.

Proposition 3.25. Let X be a BCIK-algebra. If $(X,\varphi) is$ an N-ideal of X in which φ is a θ −negative, then (X,φ) is an N-sub algebra of X.

Proof. Note that $(x*y)*x \preccurlyeq \theta*y$ for all $x,y \in X$, the θ −negativity of φ, we have $\varphi(x*y) \leq \max\{\varphi(x),\varphi(\theta*y)\} \leq \max\{\varphi(x),\varphi(y)\}$. Therefore (X,φ) is an N-sub algebra of X.

Definition 3.26. Let X be a BCIK-algebra. By a Commutative ideal of X based on φ (briefly, commutative N-ideal of X), we mean an N-structure (X,φ) in which φ satisfies and $(\forall x,y,z \in X)(\varphi(x*(y \barwedge x)) \leq \max\{\varphi((x*y)*z),\varphi(z)\})$.

Example 3.27. Consider a BCIK-algebra $X = \{\theta,a,b,c\}$. Let φ be define by

X	θ	A	b	c
φ	-0.6	-0.4	-0.3	-0.3

Routine calculations give that (X,φ) is a commutative N-ideal of X.

Theorem 3.28. Every commutative N-ideal of a BCIK-algebra X is an N-ideal of X.

Proof. Let (X,φ) be a commutative N-ideal of X. For any $x,y,z \in X$, we have $\varphi(x) = \varphi(x*(\theta \barwedge x))$ $\leq$max$\{\varphi((x*\theta)*z),\varphi(z)\} = \max\{\varphi(x*z),(z)\}$. Hence (X,φ) is an N-ideal of X.

The following example shows that the converse of Theorem 3.28. is not valid.

Example 3.29. Consider a BCIK-algebra $X = \{\theta,1,2,3,4\}$ with the following Cayley table:

$*$	θ	1	2	3	4
θ	θ	θ	θ	θ	θ
1	1	θ	1	θ	θ
2	2	2	θ	θ	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Let φ be define by

X	θ	1	2	3	4
φ	-0.7	-0.6	-0.4	-0.4	-0.4

Then (X,φ) is an N-ideal of X. But it is not a commutative N-ideal of X since $\varphi(2*(3\vec{\barwedge}2))$ = - 0.4 > - 0.7 = $\max\{\varphi((2*3)*\theta)\cdot\varphi(\theta)\}$.

Theorem 3.30. If (X,φ) is an N-ideal of a commutative BCIK-algebra X, then it is a commutative N-ideal of X.

Proof. Assume that (X,φ) is an N-ideal of a commutative BCIK-algebra X, we have

$$\left((x*(y \wedge x)) \star ((x \star y) \star z)\right)$$
$$= ((x \star (y \wedge x)) \star z) \star ((x \star y) \star z)$$

$\preccurlyeq (x*(y \wedge x))*(x*y)$

$= (x \wedge y)*(y \wedge x) = 0$

And so $\left((x*(y \wedge x))*((x*y)*z)\right)*z = \theta$, i.e., $((x*(y \wedge x)) \star ((x \star y) \star z) \preccurlyeq z$ for all $x,y,z \in X$. Since (X,φ) is an N-ideal, then $\varphi(x*(y \wedge x)) \leq \max\{\varphi((x*y)*z),\varphi(z)\}$. Hence (X,φ) is a commutative N-ideal of X.

Theorem 3.31. Let (X,φ) be an N-structure of a BCIK-algebra X and φ. Then (X,φ) is a commutative N-ideal of X if and only if it satisfies $(\forall t \in [-1,0))(C(\varphi;t) \neq \;\; \Rightarrow C(\varphi;t)$ is a commutative ideal of X).

Proof. Assume that (X,φ) is a commutative N-ideal of X. Then (X,φ) is an N-ideal of X and so every non-empty closed (φ,t)-cut $C(\varphi;t)$ $of\,\varphi$ is an ideal of X. Let $x,y,z \in X$ be such that $(x*y)*z \in C(\varphi;t) and\ z \in C(\varphi;t)$. Then $\varphi\big((x*y)*z\big) \le t$ and $\varphi(z) \le t$. It follows that $\varphi(x*(y\wedge x)) \le \max\{\varphi\big((x*y)*z\big)\cdot\varphi(z)\} \le t$. So that $x*(y\wedge x) \in C(\varphi;t)$.Hence $C(\varphi;t)$ is a commutative ideal of X.

Conversely, suppose that the condition $(\forall t \in [-1,0))(C(\varphi;t) \neq \emptyset \Rightarrow C(\varphi;t)$ is a commutative ideal of X) is valid. Obviously $\varphi(\theta) \le \varphi(x)$ for all $x \in X$. Let $\varphi\big((x*y)*z\big) = t_1$ and $\varphi(z) = t_2$ for $x,y,z \in X$. Then $(x*y)*z \in C(\varphi;t_1)$ and $z \in C(\varphi;t_2)$. Without loss of generality, we may assume that $t_1 \ge t_2$. Then $C(\varphi;t_2) \subseteq C(\varphi;t_1)$ and so $z \in C(\varphi;t_1)$. Since $C(\varphi;t_1)$ is a commutative ideal of X by hypothesis, we have $x*(y\wedge x) \in C(\varphi;t_1)$, and so $\varphi\big(x*(y\wedge x)\big) \le t_1 = \max\{t_1,t_2\} = \max\{\varphi\big((x*y)*z\big),\varphi(z)\}$. Therefore (X,φ) is a 0commutative N-ideal of X.

Corollary 3.32. If (X,φ) is a commutative N-ideal of a BCIK-algebra X, then every non-empty open (φ,t)-cut of X is a commutative ideal of X for all $t \in [-1,0)$.

Proof. Straightforward.

4. Prime ideals in BCIK-algebras

In this section.[3] S. Rethina Kumar introduce the concept of prime ideals in BCIK-algebras and we prove that this concept and the last definition of prime ideal in a lower BCIK-semi lattices are equivalent. Then we generalize some useful theorems about prime ideals on BCIK-algebras. Finally we discuss some relations between BCIK-part and prime ideals in BCIK-algebras.

Throughout this section, X is a BCIK-algebra, B is BCIK-part of X and p is p-semi simple part of X unless otherwise stated.

Definition 4.1. A proper ideal I of BCIK-algebra X is called prime if $A \cap B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ for all ideals A and B of X.

Example 4.2. Let "-" be the subtraction of integers. Then $X = \{Z,-,0\}$ is a BCIK-algebra. Clearly, $M_1 = N \cup \{0\}$ and $M_2 = \{-n | n \in N\} \cup \{0\}$ are two maximal ideal of X. Let $I \cap J \subseteq N$, If $I \nsubseteq N$ and $J \nsubseteq N$then there exist $m,n \in N$ such that $-n \in I$ and $-m \in J$. So we conclude that $-mn \in I \cap J \subseteq N \cup \{0\}$, which is impossible. Hence $N \cup \{0\}$ is a prime ideal of X. By the similar way, M_2 is a prime ideal of X.

Theorem 4.3. (i) Let I be an ideal of X. Then I is a prime ideal of X if and only if $(x) \cap (y) \subseteq I$ implies $x \in I$ or $y \in I, for\ any\ x,y \in X$.

(ii) If X is a lower BCIK-semi lattice, then are equivalent.

Proof. (i) Let I be an ideal of X such that $(x) \cap (y) \subseteq I$ implies $x \in I$ or $y \in I$. If A and are two ideal of X, such that ideals of X such that $A \cap B \subseteq I$, then there is no having in assuming $A \nsubseteq I$. Hence there exists $a \in A$ such that $a \nsubseteq I$. For any $b \in B$. Since $(a) \cap (b) \subseteq A \cap B \subseteq I$ and $a \notin I$, the primeness of I implies $b \in I$. Therefore $B \subseteq I$. Conversely, let I be a ideal of X. Clearly, $(x) \cap (y) \subseteq I$ implies $x \in I$ or $y \in I$, for any $x,y \in X$.

(ii) Since $x \cap y = x \wedge y$ for any $x,y \in X$ so are equivalent.

Clearly any prime ideal of X is an irreducible ideal. Moreover, if {0} is an irreducible ideal of X, then {0} is a prime ideal.

Definition 4.4. A nonempty subset F of X is called a finite $\cap$ −structure, if $(x \cap y) \cap F \neq \emptyset$, for all $x,y \in F$, and X is called a finite $\cap$ −structure if X\{0} is a finite $\cap$ −structure.

Proposition 4.5. Let Y be a BCIK-algebra and $f\colon X \to Y$ be an onto BCIK-homomorphism. Then the following assertions hold:

1. An ideal I of X is prime if and only if F=X-I is a finite $\cap$ −structure.
2. Let I be a closed ideal of X and J be an ideal of X containing I. If J is a prime ideal of X, then J/I is a prime ideal of X/I.
3. Let I be a prime ideal of x and $ker \subseteq I$. Then f(I) is a prime ideal of Y.
4. Let ID(X) be the set of all ideals of X, then ID(X) is a chain if and only if every proper ideal of X is prime.

Proof. (i) Let I be a prime ideal of X and $x,y \in F$. If $(x \cap y) \cap F = \emptyset$, then $(x \urcorner y) \subseteq I$. Since I is a prime ideal of X, we have $x \in I$ or $y \in I$, which is impossible. Hence $(x \cap y) \cap F \neq \emptyset$. Conversely, Let F be a finite $\cap$ −structure and $x,y \in X$ such that $(x) \cap (y) \subseteq I$. If $x \notin I$ and $y \notin I$, then $x,y \in F$ and so $(x \cap y) \cap F \neq \emptyset$. Hence, $x \cap y \nsubseteq I$, which is impossible. Therefore, $x \in I$ or $y \in I$, then I is a prime of X.

(ii) Let J be a prime ideal of X, J/I is an ideal of X/I. Let A and B be two ideals of X/I such that,$x \cap B \subseteq J/I$. There are two ideals E and F of X such that A=E/I and B=F/I. Then $(E \cap F)/I = E/(I \cap F)/I = A \cap B \subseteq J\,/\,I$. Therefore, $E \cap F \subseteq J$ and so $E \subseteq J$ or $F \subseteq J$. Hence $E/I \subseteq J/I\ orF\,/\,I \subseteq J\,/\,I$. Thus J/I is a prime ideal of X/I.

(iii)Since ker(f) is a closed ideal of X, then $x/\ker(f) \cong Y and\ I\,/\ker(f)$ is a prime ideal of $X\,/\ker f$. Moreover, $f(I) \cong I/\ker(f)$. Hence f(I) is a prime ideal of Y.

(iv)Let ID(X) be a chain and I be a proper ideal of X. Clearly $a \cap b \subseteq I$ implies $a \in I$ or $b \in I$. Hence I is a prime ideal of X. Conversely, let any proper ideal of X be prime. Let I and J be two ideals of X. Since $I \cap J$ is a proper of X, then $I \subseteq I \cap J$ or $J \subseteq I \cap J$ $and\ so$ $I \subseteq J$ or $J \subseteq I$. Therefore, ID(X) is a chain.

Corollary 4.6. Let $x \in X - \{0\}$, such that $x*y = x, for\ all\ y \in X - \{x\}$. Then there exists a prime ideal Q of X, such that $x \notin Q$.

Proof. Let $Q = X - \{x\}$. Then $0 \in Q$. If $a*b, b \in Q$, then a$\neq x$ and so $a \in Q$. Hence Q is an ideal of X. Clearly X-Q is a finite $\cap$ −structure. Q is a prime ideal of X. Therefore, there exists a prime ideal Q of X such that $x \notin Q$.

Example 4.7. Let $X - \{0,1,2,a\}$. Define the binary operation $*-$ on X by the following table:

Table 1

*	0	1	2	a
0	0	0	0	a
1	1	0	0	a
2	2	1	0	a
A	A	A	a	0

It is easy to prove that (X,∗,0) is a BCIK-algebra. Since $a * y = a, for\ any\ y \in X - \{a\}$, then $Q = X - \{a\}$ is a prime ideal of X, such that $a \notin Q$.

Proposition 4.8. Let be an ideal of X.
1. If I is a prime ideal of X, then $I/I_o is\ a\ prime\ ideal\ of\ X\ /\ I_o$.
2. If I is a closed prime ideal of X, then I_o *is* a closed prime ideal of $X\ /\ I$.
3. If I_o is a prime ideal of $X/I\ and\ I \subseteq B$, then I is a prime ideal of X.

Proof. (i) Since I_o is a closed ideal of X, then $I/I_o is$ an ideal of $X\ /\ I_o$. Let $A'\ and\ B' be\ two\ ideals\ of\ X\ /\ I_o$ such that $A' \cap B^i \subseteq I\ /\ I_o$. Then, there are ideals A and B of X containing I_o such that $A' = A\ /\ I_o$ and$B' = B\ /\ I_o$ and so $(A \cap B)/I_o = A' \cap B' \subseteq I\ /\ I_o$. Hence $A \cap B \subseteq I\ ans\ so\ A \subseteq I\ or\ B \subseteq I$ and so $B' \subseteq I\ /\ I_o$. Therefore, $I\ /\ I_o$ is a prime ideal of $X\ /\ I_o$.

(ii)If I is closed, then $I = I_o$ and so $X/I = X/I_0$ and $I/I_o = I_o$.

(iii)Let $I \subseteq B$ and I_o be a prime ideal of $X\ /\ I$ and $x \cap y \subseteq I$ for some $x, y \in X$. If $I_o \in I_x \cap I_y$. There exist $n, m \in N\ such\ that\ I_o * (I_x)^n = I_o\ and\ I_o * (I_y)^m = I_o$and so ∗on X/I we get $I_{u*x^n} = I_u * I_{x^n} = I_o$ and $I_{u*y^y} = I_o * I_{y^m} = I_o$, it follows that $u * x^n \in I\ and\ u * y^m \in I\ and\ so\ u * x^m = a. u * y^m = b$. For some $a, b \in I$. Since $I \subseteq B$ then we obtained $(u * a) * b \in x \cap y and\ so\ (u * a) * b \in I$. Moreover, I is an ideal and $a, b \in I$. Hence $u, 0 * u \in I\ and\ so\ I_o = I_0$. Thus $I_x \cap I_y \subseteq I_o$. Since I_o is a prime ideal of X/I, then we have $I_x = I_o\ or\ I_y = I_0\ and\ so\ x \in I\ or\ y \in I$. Hence I is a ideal of X.

By definition of prime and irreducible ideal, any prime ideal is an irreducible ideal in any BCIK-algebra. But the converse is false. In next example, we will show that there exists an irreducible ideal which is not prime.

Example 4.9. (i) Let X={0,a,b,c}. Define the binary operation $a * x$ on X by the following table:

Table 2

∗	0	a	b	c
0	0	a	b	c
A	A	0	c	b
B	B	c	0	a
C	C	b	a	0

Then $(x,*,0)$ is a BCIK-algebra and {{0},{0,a},{0,b} and {o,c}} is the set of all proper ideals of X. Clearly, {0,a},{0,b} and {0,c} are irreducible of X. We have $\{0, a\} \cap \{0, b\} \subseteq \{0, c\}$. Hence {0,c} is not a prime ideal of X. By similar way, {0}, {0,a}, {0,b} are not prime ideal of X. Therefore, X has not any prime ideal.

(ii)Let $(X,*,0)$ be the BCIK-algebra . thenI={0,a} is an irreducible ideal of X. Now, we have $b, c \in X - I\ and\ b \cap c = \{0, b\} \cap \{0, c\} = \{0\}$ and so $(b \cap c \cap (X - I)) = \emptyset$. Therefore, X-I is not a finite∩-structure.

(iii)Let X={0,1,a,b,c}. Define the binary operation $a * u\ on\ X$ by the following table:

Table 3

∗	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	A	A	0	c	b
c	B	B	c	0	c
c	C	C	b	a	0

Then $(X,*,0)$ is a BCIK-algebra and {{0},{0,1a},{0,1,b},{0,1,c}} is the set of all proper ideals of X and $\{0,1, b\} \cap \{0,1, c\} \subseteq \{0,1, a\}$ and so $I = \{0,1, a\}$ is not a prime ideal of X. But, $\{\{I_0\}, \{I_o, I_a\}\}$ is the set of all ideals of X/I. Therefore, is not true in general.

Theorem 4.10. Let A be an ideal of X such that $A \subseteq B$. Then $I \cap J \subseteq A$ if and only if $(A \cup I) \cap (A \cup J) = A$, for any ideals I and J of X.

Proof. Let $(A \cup I) \cap (A \cup J) = A$. Since $I \cap J \subseteq A$. Clearly, $A \subseteq (A \cup I) \cap (A \cup J)$. Let $u \in (A \cup I) \cap (A \cup J)$. Since A is an ideal of X, then we get $((\ldots. (u * x_1) * \ldots) * x_n) \in A$, foe some $n \in N$ and $x_1, \ldots x_n \in I$. It follows that, there exists $m_1 \in A$, such that $\big((\ldots. (u * x_1) * \ldots) * x_n\big) \in m_1$. By the similar way, we have $\big((\ldots. (u * y_1) * \ldots) * y_m\big) \in m_2$, for some $m \in N$, $y_1, \ldots y_n \in J$ and $m_2 \in A$. Hence by, we get $(((\ldots. (u * m_1) * \ldots) * x_n)) * x_1 - ((\ldots (u * x_1) * \ldots) * x_n)) * m_1 - 0$. Since I is an ideal of X and $x_1, \ldots x_n \in I$, then $u * m_2 \in$ J. Since m_1, m_2 ∈B, we conclude that$(u * m_1) * m_2 \leq u * m_1 and\ (u * m_1) * m_2 \leq u * m_2, and\ so\ (u * m_1) * m_2 \in I \cap J \subseteq A$. Hence $m \in A$ and so $(A \cup B)(A \cup J) \subseteq A$. Therefore, $(A \cup I) \cap (A \cup J) = A$.

Example 4.11. Let $(X,*,0)$ be the BCIK-algebra. Then $I = \{0, a\}, J - \{0, b\} and\ K - \{0, c\}$ are three ideals of X $and\ J \cap K \subseteq I$, but $(I \cup J) - X - (I \cup K)$. Hence, if A is not contained in B then may not true, in general.

Remark 4.12. We know that, if M is a maximal ideal of lower BCIK-semi lattice X, then M is a prime ideal, then any maximal ideal is a prime ideal in any BCIK-algebra.

Theorem 4.13. If M is a maximal ideal of BCIK-algebra X, then M is a prime ideal of X.

Proof. Let $x \cap y \subseteq M, for\ some\ x, y \in X. If\ x \notin M\ and\ y \notin M, then\ M \cup \{x\} = X\ and\ M \cup \{y\} = X\ and\ so\ (M \cup \{x\}) \cap (M \cup \{y\}) = X$. Now, $x \cap y \not\subset M$, which is impossible, then M is a prime ideal of X.

Example 4.14. Let X be the BCIK- algebra. Clearly, $M - \{0, b\}$ is a maximal ideal of X. Since$\{0, a\} \cap \{0, c\} = \{0\} \subseteq M. \{0, a\} \not\subseteq M and\ \{0, c\} \not\subseteq M$, then M is not a prime ideal of X. It has been known, if X is a lower BCIK-semi lattices and A is an ideal of X such that $A \cap F - \emptyset$, where F is $\wedge -$closed subset of X. Then there is a prime ideal Q of X such that $A \subseteq Q$ and $Q \cap F = \emptyset$. I generalize this theorem for BCIK-algebra.

Theorem 4.15. Let X be a BCIK-algebra and F be a nonempty subset of X such that F is closed under "∘" . where $x \circ y := x * (x * y), for\ any\ x, y \in F$. If A is an ideal of X such that $A \cap F - \emptyset$, then there exist a prime ideal Q of X such that $A \subseteq Q\ and\ Q \cap F = \emptyset$.

Proof. Let $S = \{I | I \ \lhd X, A \subseteq I\ and\ F \cap I = \emptyset\}$. Then S with respect to the inclusion relation "$\subseteq$" forms a poset. Clearly, every chain on S has an upper bound (unionof its elements). Hence, S has a maximal element , say Q. Obviously, Q is an ideal of X such that $P \cap A = \emptyset$. we claim that Q is a prime ideal, otherwise there are I,J of X, such that $I \cap J \subseteq Q, I \nsubseteq Q$ and $J \nsubseteq Q$. By maximality of Q we have $(Q \cup I) \cap F \neq \emptyset$ and $(Q \cap J) \cap F \neq \emptyset$. Let $a \in (Q \cup I) \cup F$ and $b \in (Q \cup J) \cap F$. Since $(a \circ b) * b = 0$, we have $a \circ b \in ((Q \cup I) \cap (Q \cup J))$. On the other hand a,b$\in F$ and F is $\circ$−closed and so $a \circ b \in F$. Hence $a \circ b \in ((Q \cup I) \cap (Q \cup J) \cap F$. Comparison of last relation with $Q \cap F = \emptyset$ gives $Q \neq (Q \cup I) \cap (Q \cup J)$. Hence $I \cap J \nsubseteq Q$. Therefore, Q is a prime ideal.

Corollary 4.17. Let X be a BCIK-algebra. Then the following assertions hold:

1. For any $x \in X\{0\}$, there exists a prime ideals Q such that $x \notin P$.
2. $\cap\{Q\}|Q$ is a prime ideal of $x\} - \{0\}$.
3. Any proper ideal A of X can be expressed as the intersection of all prime ideals of X containing A.
4. Let Y be a BCIK-algebra and $f: X \longrightarrow Y$ be a BCIK-homomorphism, such that f(X) is an ideal of Y. If I is a prime ideal of Y and $f^{-1}(I) \neq X$, then $f^{-1}(I)$ is a prime ideal of X.

Proof. (i) Let $x \in X\{0\}$. Then we set $A = \{0\}$ and $F = \{x\}$. Clearly, F is $\circ$−closed and $A \cap F = \emptyset$. Hence, there exists a prime ideal Q such that Q is not contain x.

(ii) The proof is straightforward.

(iii)Let $a \in (X - A)$ and F={a}. Then (BCIK), $x * (x * y) \in F$, for all $x, y \in F$. There exists a prime ideal Q_n of X such that $a \notin Q_a$ and $A \subseteq Q_a$. Therefore. $A \subseteq {}_{a\in X-A}\cap Q_a$. On the other hand $b \notin {}_{a\in X-A}\cap Q_a$ for any $b \in X$ −A. Hence ${}_{a\in X-A}\cap Q_a$ ⊆A and so A $= {}_{a\in X-A}\cap Q_a$.

(iv)Let $x \cap y \subseteq f^{-1}(I)$, for some $x, y \in X$. If $(f(x) \cap (f(y)) = 0$ then $(f(x) \cap (f(y)) \subseteq I$. Let $u \in (f(x) \cap (f(x)) - \{0\}$. Then there exist $m, n \in N$ such that $u * f(x)^n = 0$ and $u * f(y)^m = 0$. Since f(X) is an ideal of Y and $(f(x)) \subseteq f(X), (f(y)) \subseteq f(X)$, then $u = f(a)$ for some $a \in X$. Moreover, f is a BCIK-homomorphism and so $f(a * x^n) = 0 = f(a * y^m)$. Hence, $a * x^n \in f^{-1}(I)$ and $a * y^n \in f^{-1}(I)$ and so $a \in (f^{-1}(I) \cup \{x\} \cap (f^{-1}(I) \cup \{y\})$. Since $x \cap y \subseteq f^{-1}(I)$, then $a \in f^{-1}(I)$ and so $u = f(a) \in I$. Hence $(f(x)) \cap (f(y)) \subseteq I$. Now, since I is a prime ideal of Y we have $f(x) \in I$ or $f(y) \in I$ and so $x \in f^{-1}(I)$ or $y \in f^{-1}(I)$. Therefore $f^{-1}(I)$ is a prime ideal of X.

Corollary 4.18. Let A be an ideal of X generated by P. If I is a proper ideals of X containing P, then $I = \cap\{\cup\{A_x | A_x \subset J\} | J$ is a prime ideal of $X \backslash A\}$.

Proof. Clearly, X/A is a BCIK-algebra. We have I/A=$\cap\{J\ J$ is a prime ideal of X/A}. Let J be a prime ideal of X/A. Since $A = P = (P \cup P) = P + P$, then A is a closed ideal of X and so $J = F_J$/A, where $F_J = \cup\{A_x | A_x \in J\}$. Therefore, I/A$\cap\{F_J/A\ |J$ is a prime ideal of $X/A\} = (\cap \{F_J|J$ is a prime ideal of $X/A\}) / A$ Now, we conclude that $I = \cap\{F_J|J$ is a prime ideal of $X/A\}$.

Let X be lower BCIK-semi lattice and I be an ideal of X. If X/I is a BCIK-chain then I is a prime ideal of X, if X has not any prime ideal we say the intersection of all prime ideals of X is X.

Theorem 4.19. Let X be a BCIK-algebra and I be a prime ideal of X.

1. If $I \subseteq B$ and ID(X/I) is a chain, then I is a prime ideal of X.
2. Let $M_1, \ldots, M_n$ and M be maximal ideals of X such that $\cap_{i-1}^{n} M_i \subseteq M$. Then there exists $j \in \{1,2, \ldots, n\}$, such that $M_J = M$.
3. Let X be a nonzero nilpotent BCIK-algebra and $S = \{P_\propto | \propto \in J\}$ be the set of prime ideals of X. Then $\cap_{\propto \in J} P_\propto = \{0\}$ if and only if X is sub direct product of special family $\{X_i\}_{i\in I}$, such that X_i is a finite $\cap$ −structure, for any $i \in I$.

Proof. (i) Let $x, y \in X$ such that $x \cap y \in X$. Since ID(X/I) is a chain, then $I_x \subseteq I_y$ or $I_y \subseteq I_x$. Let $I_x \subseteq I_y$, there exist $n \in N$ such that $I_{x*y^a} = I_x * I_{y^a} = I_x * (I_y)^n = I_0$ and so $x * y^n \in I$. Since $I \subseteq B$, then we have $x * (x * y^n) \in x \cap y$ and so $x \in I$. By the similar way, we get $y \in I$, when $I_y \subseteq I_x$. Therefore, I is a prime ideal of X.

(ii) M is a prime ideal of X. Hence there exists $J \in \{1, \ldots, n\}$ such that $M_j \subseteq M$. Since M_j is a maximal ideal of X we obtain that $M_j = M$.

(iii)Clearly, the map $\varphi: X \to \coprod_{\alpha \in J} X / P_\propto$define by $\varphi(x) = ((P_a)x)_{\propto \in J}$, for all $x \in X$, is a homomorphism and $\ker(\varphi) = \coprod_{\propto \in J} P_\propto = \{0\}$. Thus φ is a one to one homomorphism and so it is a sub direct embedding. Now, let $\propto \in J$. Since X is nilpotent, then I is closed and so$(P_\propto)_a$ is a prime ideal of X/P. Hence X/$P_\propto - \{P_0\}$ is a finite $\cap$ −structure. Conversely, Let X be sub direct product of family $\{X_i\}_{i\in I}$, such that$X_i$is a finite $\cap$ −structure for any $i \in I$. Then there is an one to one BCIK-homomorphism $\varphi: X \prod_{j\in J} X_j$ such that $(\pi_i \circ \varphi): X \neg X_i$ is an onto BCIK-homomorphism and so X/$B_i \cong X_i$, for any $i \in J$, where $B_i = (\pi_i \circ \varphi)^{-1}(\{0\})$. Let $i \in J$. Since X_i is a $\cap$ −structure, then X/B_i is finite $\cap$ −structure, B_i is a prime ideal of X. Clearly $\prod_{j\in J} B_j = \ker(\varphi) = \{0\}$. Therefore, the intersection of all prime ideals of x is {0}.

Corollary 4.20. Every non zero BCIK-algebra is sub direct product of a family of finite $\cap$ −structure BCIK-algebra.

Example 4.21. Let X={0,1,2,a,b}. Define the binary operation $a * u$ on X by the following table:

Table 4

*	0	1	2	a	B
0	0	0	0	b	A
1	1	0	1	b	A
2	2	2	0	b	A
A	A	A	a	0	B
B	B	B	b	a	0

Then$(X,*,0)$ is a BCIK-algebra. Let $I = \{0,1\}$. Then $I \subseteq B$ and $\{\{I_0\}, \{I_0, I_2\}, X/I\}$ is the set of all ideals of X/I. Therefore, the set of ideals of X/I is a chain, we conclude that I is a prime ideal of X.

Note 4.22. Let X be a p-semi simple BCIK-algebra. Then $(X,\cdot,0)$ is an Abelian group, where $x \cdot y = x * (0 * y)$ for all $x, y \in X$. Moreover, any closed ideal of X is a subgroup of $(X,\cdot,0)$.

Theorem 4.23. Let X be an associative BCIK-algebra and I be an ideal of X.

1. If there exist distinct element x, y of X such that $x, y \notin J$. Then I is not a prime ideal.

2. If X is of order n>2, then there is not any prime ideal on X.

Proof. (i) Since X is an associative BCIK-algebra, we have $x = \{x, 0\} and\ y = \{0, y\} and\ so\ x \cap y = 0$. Therefore, I is not a prime ideal of X.

(ii)Let I be a proper ideal of X. Since X is finite, then I is a closed ideal. Hence I is a subgroup of $(X,\cdot,0)$ and so there exists I∈ $N - \{1\}$ such that n=I|I|, where |I| is the number of elements of I. Hence |I|≤ $n - 2$. Now I is not a prime ideal of X and so X has not any prime ideals.

Theorem 4.24. Let M be a maximal ideal of X containing P. If I=P, then M/I is a prime ideal of X/I.

Proof. Since, I=P=PUP=P+P. then I is a closed ideal of X. Since $P \subseteq M$, we have $I \subseteq M$ and so M/I is a maximal ideal of X/I is a maximal ideal of X/I is a BCIK-algebra. Hence M/I is a prime ideal of X/I.

Example 4.25. Let X={0,1,a,b}. Define the " $*$ " on X by

Table 5

*	0	1	a	B
0	0	0	a	A
1	1	0	b	A
A	A	a	0	0
B	B	A	1	0

Clearly,$(X,*,0)$ is a BCIK-algebra and {{0},{0,a},{0,1}} is the set of all proper ideals of X. Hence M={0,a} is a maximal ideal of X. It is obvious that P={0,a} is the p-semi simple part of X. M/I is a prime ideal of X/I, where I=P.

Lemma 3.16. Let X be a nilpotent BCIK-algebra. Then for any $b \in B \setminus \{0\}$, there exists a prime ideal Q such that $b \notin Q$.

Proof. Let $b \in B \setminus \{0\}$, there exists a prime ideal I of B such that $b \notin I$. Let P be p-semi simple part of X. we claim that $b \notin I + P$. Otherwise $b \in I + P$. There exist $a_1, \dots, a_n \in I$ such that $(\dots(b * a_1) * \dots) * a_n \in P$. Since B is a closed ideal of X we have $(\dots(b * a_1) * \dots) * a_n \in B$. Hence$(\dots(b * a_1) * \dots) * a_n \in B \cap P = \{0\}$. Therefore, $(\dots(b * a_1) * \dots) * a_n = 0 \in I$. Since I is an ideal of X containing $a_1, \dots, a_n$, we conclude that $b \in I$, which is a contradiction. Hence $b \notin I + P$. It remains to show that I+P is a prime ideal of X. Let J and K be two ideals of X such that $J \cup K \subseteq I + P$. Then $(J \cap B) \cap (K \cap B) = (J \cap K) \cap B \subseteq (I + P) \cap B$. Now we show that (I+P)$\cap B = I$. Clearly, I⊆ $(I + P) \cap B$. Let $x \in (I + P) \cap B$. Then there exist $a_1, \dots, a_n \in I$ such that $(\dots(x * a_1) * \dots) * a_n \in P$. Since x.$a_1, \dots, a_n \in B$ we have $(\dots(x * a_1) * \dots) * a_n \in P \cap B = \{0\}$. Moreover, Since $a_1, \dots, a_n \in I$ we obtain $x \in I$. Hence (I+P)$\cap B \subseteq I$. Therefore (I+P)$\cap B = I$ we have $(J \cap B) \cap (K \cap B) \subseteq I$. Since I is a prime ideal of B we have $J \cap B \subseteq I\ or\ K \cap B \subseteq I$. Assume that $J \cap B \subseteq I$. Since $x * (0 * (0 * x)) \in B$ and X is nilpotent , $x * \left(0 * (0 * x)\right) \in B \cap J\ and\ 0 * (0 * x) \in P$ for all $x \in J$. Since $J \cap B \subseteq I$ we have $x \in I + P$. Therefore, $J \subseteq I + P$, If $K \cap B \subseteq I$, then by the similar way, we obtain $K \subseteq I + P$, we obtain $J \subseteq I + P\ or\ K \subseteq I + P$. Hence I+P is a prime ideal of X.

Corollary 4.27. If X is a nilpotent BCIK-algebra such that B≠ {0}, then X has a prime ideal.

Theorem 4.28. Let X be a nilpotent BCIK-algebra.
1. For any $x \in X - P$, there exists a prime ideal Q of X, such that $x \notin Q$.
2. $\cap \{Q | Q\ is\ a\ prime\ ideal\ of\ X\} \subseteq P$.

Proof. (i) Let $x \in X - P$. Then BCIK-algebra we conclude that $x * \left(0 * (0 * x)\right) \in B - \{0\}$. Hence, there is a prime ideal Q of X such that $x * (0 * (0 * x)) \notin Q$. Therefore, $x \notin Q$. Since if $x \in Q$, then by BCIK-algebra we get $x * \left(0 * (0 * x)\right) * x = 0 * x \in Q$ (since Q is closed) and so $x * \left(0 * (0 * x)\right) \in Q$, Which is impossible.

(ii)It is straight consequent of (i).

5. Translations of N-sub algebras and N-ideals.

For any N-function φ on X, we denote $\perp := -1 - \inf\{\varphi(x) | x \in X\}$. For any $\propto \in [\perp, 0]$, we define $\varphi_\alpha{}^T(x) = \varphi(x) + \alpha$ for all $x \in X$. Obviously, $\varphi_\alpha{}^T$ is a mapping from X to [-1,0] that is , $\varphi_\alpha{}^T$ is an N-function on X. we say that (X,$\varphi_\alpha{}^T$) is an $\propto$ −translation of (X, φ).

Theorem 5.1. For every $\propto \in [\perp, 0]$, the $\propto$ −translation (X,$\varphi_\alpha{}^T$) of an N-sub algebra (resp. N-ideal) (X, φ) is an N-sub algebra (resp. N-ideal) of X.

Proof. For any $x, y \in X$, we have

$$\varphi_\alpha{}^T(x * y) = \varphi(x * y) + \alpha \leq \max\{\varphi(x), \varphi(y)\} + \alpha$$

$$\max\{\varphi(x) + \alpha, \varphi(y) + \alpha\} = \max\{\varphi_\alpha{}^T(x), \varphi_\alpha{}^T(y)\}.$$

Therefore (X,$\varphi_\alpha{}^T$) is an N-sub algebra of X. Let $x, y \in X$. Then $\varphi_\alpha{}^T(\theta) = \varphi(\theta) + \alpha \leq \varphi(x) + \alpha = \varphi_\alpha{}^T(x)$, and

$$\varphi_\alpha{}^T(x) = \varphi(x) + \propto \leq \max\{\varphi(x * y), \varphi(y)\} + \alpha$$

$$= \max\{\varphi(x * y) + \alpha, \varphi(y) + \alpha\} = \max\{\varphi_\alpha{}^T(x * y), \varphi_\alpha{}^T(y)\}.$$

Hence (X,$\varphi_\alpha{}^T$) is an N-ideal of X.

Theorem 5.2. If there exists $\propto \in [\perp, 0]$ such that the $\propto$ −translation (X,$\varphi_\alpha{}^T$) of (X, φ) is an N-sub algebra (resp. N-ideal) of X, then (X, φ) is an N-sub algebra (resp. N-ideal) of X.

Proof. Assume that (X,$\varphi_\alpha{}^T$) is an N-sub algebra (resp. N-ideal) of X for some $\propto \in [\perp, 0]$. Let $x, y \in X$. Then

$$\varphi(x * y) + \alpha = \varphi_\alpha{}^T(x * y) \leq \max\{\varphi_\alpha{}^T(x), \varphi_\alpha{}^T(y)\}$$

$$= \max\{\varphi(x) + \alpha, \varphi(y) + \alpha\} = \max\{\varphi(x), \varphi(y)\} + \alpha$$

Which implies that $\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}$. Therefore (X, φ) is an N-sub algebra of X. Now suppose that there exists$\propto \in [\perp, 0]$such that (X,$\varphi_\alpha{}^T$) is an N-ideal of X. Let $x, y \in X$. Then $\varphi(\theta) + \alpha = \varphi_\alpha{}^T(\theta) \leq \varphi_\alpha{}^T(x) = \varphi(x) + \alpha$, and so $\varphi(\theta) \leq \varphi(x)$. Finally,

$$\varphi(x) + \alpha = \varphi_\alpha{}^T(x) \leq \max\left(\varphi_\alpha{}^T(x * y), \varphi_\alpha{}^T(y)\right\}$$

$$= \max\{\varphi(x * y) + \alpha, \varphi(y) + \alpha\} = \max\{\varphi(x * y), \varphi(y)\} + \alpha,$$

Which implies that $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}$. Thus (X, φ) is an N-ideal of X.

For any N-function φ on X, $\propto \in [\perp, 0]$ and $t \in [-1, \propto)$, let $L_\propto(\varphi; t) := \{x \in X | \varphi(x) \leq t - \propto\}$.

Proposition 5.3. Let (X, φ) be an N-structure of X and φ, and let $\propto \in [\perp, 0]$. If (X, φ) is an N-sub algebra (resp. N-ideal) of X, then $L_\propto(\varphi; t)$ is a sub algebra (resp. ideal) of X for all $t \in [-1, \propto)$.

Proof. Assume that (X, φ) is an N-sub algebra of X. Let $x, y \in L_\propto(\varphi; t)$. Then $\varphi(x) \leq t - \propto$ and $\varphi(y) \leq t - \propto$ Thus

$\varphi(x*y) \le \max\{\varphi(x), \varphi(y)\}$, and hence $x \in L_\propto(\varphi;t)$. Clearly $\theta \in L_\propto(\varphi;t)$. Therefore $L_\propto(\varphi;t)$ is an ideal of X.

If we do not give a condition that (X,φ) is an N-sub algebra (resp. N-ideal) of X then $L_\propto(\varphi;t)$ may not be a sub algebra (resp. ideal) of X as seen in the following example.

Example 5.4. Consider a BCIK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	θ	θ
b	a	a	θ	θ	θ
c	c	a	a	θ	θ
d	d	c	c	a	θ

Define an N-function φ on X by

X	θ	a	b	c	d
φ	-0.7	-0.4	-0.6	-0.3	-0.5

Then $\perp = -0.3$ and (X,φ) is not an N-sub algebra of X because $\varphi(d*b) = \varphi(c) = -0.3 > -0.5 = \max\{\varphi(d), \varphi(b)\}$ For $\propto = -0.1 \in [-0.3, 0]$ and $t = -0.5$, we obtain $L_\propto(\varphi;t) = \{\theta, a, b, d\}$ which is not a sub algebra of X since $d*b = c \notin L_\propto(\varphi;t)$.

Example 5.5. Consider a BCIK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:

$*$	θ	a	b	c	d
θ	θ	d	c	b	a
a	a	θ	d	c	b
b	b	a	θ	d	c
c	c	b	a	θ	d
d	d	c	b	a	θ

Define an N-function φ on X by

X	θ	a	b	c	d
φ	-0.6	-0.5	-0.6	-0.3	-0.2

Then $\perp = -0.4$ and (X,φ) is not an N-ideal of X since $\varphi(d) = -0.2 > -0.6 = \max\{\varphi(d*b), \varphi(b)\}$ For $\propto = -0.15 \in [\perp, 0]$ *and* $t = -0.5$ *we have* $L_\propto(\varphi;t) = \{0, a, b\}$ which is not an ideal of X since $c*b = a \in L_\propto(\varphi;t)$ and c$\notin L_\propto(\varphi;t)$.

Theorem 5.6. Let (X,φ) be an N-structure and $\propto \in [\perp, 0]$. Then the $\propto$ −translation $(X, {\varphi_\alpha}^T)$ of (X,φ) is an N-sub algebra (resp. N-ideal) of X if and only if $L_\propto(\varphi;t)$ is a sub algebra (resp. ideal) of X for all $t \in [-1, \propto]$.

Proof. Assume that $(X, {\varphi_\alpha}^T)$ is an N-sub algebra of X. Let $x, y \in L_\propto(\varphi;t)$. Then $\varphi(x) \le t - \propto$ *and* $\varphi(y) \le t - \propto$. Hence

$$\varphi(x*y) + \alpha = {\varphi_\alpha}^T(x*y) = \max({\varphi_\alpha}^T(x), {\varphi_\alpha}^T(y))$$

$$= \max\{\varphi(x)+\alpha, \varphi(y)+\alpha\} = \max\{\varphi(x), \varphi(y)\} + \alpha$$

$$\le t - \propto + \propto = t$$

And so $\varphi(x*y) \le t - \propto$ i.e., $x*y \in L_\propto(\varphi;t)$. Therefore $L_\propto(\varphi;t)$ is a sub algebra of X. Suppose that $L_\propto(\varphi;t)$is a sub algebra of X for all $t \in [-1, \propto]$. We claim that ${\varphi_\alpha}^T(x*y) = \max({\varphi_\alpha}^T(x), {\varphi_\alpha}^T(y))$ for all $x, y \in X$. If it is not valid, then ${\varphi_\alpha}^T(a*b) > s \ge \max({\varphi_\alpha}^T(a), {\varphi_\alpha}^T(b))$ for some $a, b \in X$ and $s \in [-1, \propto]$. It follows that $\varphi(a) \le s - \propto$ and $\varphi(b) \le s - \propto$, but $\varphi(a*b) > s - \propto$. Thus $a \in L_\propto(\varphi;s)$ and $b \in L_\propto(\varphi;s)$, but $a*b \notin L_\propto(\varphi;s)$. This is a contradiction and therefore $(X, {\varphi_\alpha}^T)$ is an N-sub algebra of X, suppose that $(X, {\varphi_\alpha}^T)$ is an N-ideal of X. Let $t \in [-1, \propto]$. For any $x \in L_\propto(\varphi;t)$, we prove $\varphi(\theta) \le \varphi(x) \le t - \propto$ and thus $\theta \in L_\propto(\varphi;t)$ Let $x, y \in X$ be such that $x, y \in L_\propto(\varphi;t)$ and $y \in L_\propto(\varphi;t)$. Then $\varphi(x*y) \le t - \propto$ and $\varphi(y) \le t - \propto$, i.e., ${\varphi_\alpha}^T(x*y) \le t$ and ${\varphi_\alpha}^T \le t$. It follows that $\varphi(x) + \alpha = {\varphi_\alpha}^T(x) \le \max\{{\varphi_\alpha}^T(x*y), {\varphi_\alpha}^T(y)\} \le t$ *so that* $\varphi(x) \le t - \propto$, i.e., $x \in L_\propto(\varphi;t)$. Hence $L_\propto(\varphi;t)$ is an ideal of X. Finally assume that $L_\propto(\varphi;t)$ is an ideal of X for all $t \in [-1, \propto]$. We claim that

1. ${\varphi_\alpha}^T(\theta) \le {\varphi_\alpha}^T(x)$ *for all* $x \in X$.
2. ${\varphi_\alpha}^T(x) \le \max\{{\varphi_\alpha}^T(x*y), {\varphi_\alpha}^T(y)\}$ *for all* $x, y \in X$.

If (i) is not valid, then ${\varphi_\alpha}^T(\theta) > s_0 \ge {\varphi_\alpha}^T(a)$ for some $a \in X$ and $s_0 \in [-1, \propto]$. Thus $\varphi(a) + \alpha = {\varphi_\alpha}^T(a) \le s_0$, i.e., $\varphi(a) \le s_0 - \propto$, and $\varphi(\theta) + \alpha = {\varphi_\alpha}^T(\theta) > s_0, i.e., \varphi(\theta) > s_0 - \propto$. Therefore $a \in L_\propto(\varphi;s_0)$, but $\theta \notin L_\propto(\varphi;s_0)$, which is a contradiction. If (ii) is not true, then

${\varphi_\alpha}^T(a) > s_1 \ge \max\{{\varphi_\alpha}^T(a*b), {\varphi_\alpha}^T(b)\}$ for some $a, b \in X$ and $s_1 \in [-1, \propto]$. It follows that $\varphi(a*b) + \alpha = {\varphi_\alpha}^T(a*b) \le s_1, \varphi(b) + \alpha = {\varphi_\alpha}^T(b) \le s_1 and\ \varphi(a) + \alpha = {\varphi_\alpha}^T(a) > s_1$ so that $a*b \in L_\propto(\varphi;s_1)$ and b$\in L_\propto(\varphi;s_1)$, but a$\notin L_\propto(\varphi;s_1)$. This is a contradiction. Consequently$(X, {\varphi_\alpha}^T)$is an N-ideal of X.

For any N-functions φ and ω, we say that (X,ω) is a retrenchment of (X,φ) if $\omega(x) \le \varphi(x)$ for all $x \in X$.

Definition 5.7. Let φ and ω be N-functions on X. We say that (X,ω) is a relrenched N-sub algebra (resp. relrenched N-ideal) of (X,φ) if the following assertions are valid:

1. (X,ω) is a retrenchment of (X,φ).
2. *If* (X,φ) is an N-sub algebra (resp. N-ideal) of X, then (X,ω) is an N-sub algebra (resp. N-ideal) of X.

Theorem 5.8. Let (X,φ) be an N-sub algebra (resp. N-ideal) of X. For every$\propto \in [\perp, 0]$, the $\propto$ −translation $(X, {\varphi_\alpha}^T)$ and (X,φ) is a retrenched N-sub algebra (resp. retrenched N-ideal) of (X,φ) .

Proof. Obviously, $(X, {\varphi_\alpha}^T)$ is a retrenchment of (X,φ). We conclude that $(X, {\varphi_\alpha}^T)$ is a retrenched N-sub algebra (resp. retrenched n-ideal) of (X,φ).

The converse of Theorem 5.8. is not true as seen in the following example.

Example 5.9. Consider a BCIK-algebra $X = \{0, a, b, c, d\}$ *with the following Cayley table:*

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	a	θ	θ
b	b	b	θ	b	θ
c	c	a	c	θ	a
d	d	d	d	d	θ

Define N-functions $\varphi_1$and φ_2 on X by

X	θ	a	b	c	d
φ_1	-0.9	-0.6	-0.4	-0.7	-0.3
φ_2	-0.8	-0.4	-0.6	-0.4	-0.1

Then (X,φ_1) is an N-sub algebra of X, and (X,φ_2) is an N-ideal of X. Let ω_1 and ω_2 be N-functions on X defined by

X	θ	a	b	c	d
ω_1	-0.92	-0.65	-0.43	-0.71	-0.38
ω_2	-0.88	-0.45	-0.63	-0.45	-0.19

Then (X,ω_1) is a retrenched N-sub algebra of (X,φ_1), which is not an $\propto$-translation of (X,φ_1) for $\propto \in [\perp, 0]$. Also,

(X, ω_2) is a retrenched N-ideal of (X, φ_2), which is not an $\propto$-translation of (X, φ_2) for $\propto \in [\perp, 0]$.

For two N-structures (X, φ_1) and (X, φ_2), we define the union $\varphi_1 \cup \varphi_2$ and the intersection $\varphi_1 \cap \varphi_2$ of φ_1 and φ_2 as follows:

$(\forall x \in X)\big((\varphi_1 \cup \varphi_2)(x) = \max\{\varphi_1(x) \cdot \varphi_2(x)\}\big).$

$(\forall x \in X)\big((\varphi_1 \cap \varphi_2)(x) = \min\{\varphi_1(x) \cdot \varphi_2(x)\}\big).$

Respectively, Obviously, $(X, \varphi_1 \cup \varphi_2)$ and $(X, \varphi_1 \cap \varphi_2)$ are N-structures which are called the union and the intersection of (X, φ_1) and (X, φ_2), respectively.

Lemma 4.10. If (X, φ_1) and (X, φ_2) are N-sub algebra (resp. N-ideals) of X, then the union $(X, \varphi_1 \cup \varphi_2)$ is an N-sub algebra (resp. N-ideal) of X.

Proof. Straightforward.

Example 5.11. Consider a BCIK-algebra $X = \{\theta, 1, 2, a, b\}$ with the following Cayley table:

$*$	θ	1	2	a	b
θ	θ	θ	θ	b	a
1	1	θ	1	b	a
2	2	2	θ	b	a
a	a	a	a	θ	b
b	b	b	b	a	θ

Define N-function φ_1 and φ_2 on X by

X	θ	1	2	a	b
φ_1	-0.7	-0.2	-0.2	-0.5	-0.4
φ_2	-0.9	-0.6	-0.7	-0.3	-0.3

Then (X, φ_1) is an N-sub algebra of X, and (X, φ_2) is an N-ideal of X which is also an N-sub algebra of X. But (X, φ_1) is not an N-ideal of X since $\varphi(2) = -0.2 > -0.4 = \max\{\varphi(2 * a) \cdot \varphi(a)\}$. The union $\varphi_1 \cup \varphi_2$ and the intersection $\varphi_1 \cap \varphi_2$ are given by

X	θ	1	2	a	b
$\varphi_1 \cup \varphi_2$	-0.7	-0.2	-0.2	-0.3	-0.3
$\varphi_1 \cap \varphi_2$	-0.9	-0.6	-0.7	-0.5	-0.4

Then $(X, \varphi_1 \cup \varphi_2)$ is an N-sub algebra of X, but it is not an N-ideal of X because $(\varphi_1 \cup \varphi_2)(1) = -0.2 > -0.3 = \max\{(\varphi_1 \cup \varphi_2)(1 * b).(\varphi_1 \cup \varphi_2)(b)\}$. This shows that the union of an N-algebra and an N-ideal may not be an N-ideal. We see that

$(\varphi_1 \cap \varphi_2)(1 * a) = (\varphi_1 \cap \varphi_2)(b) = -0.4 > -0.5$

$= \max\{(\varphi_1 \cap \varphi_2)(1), (\varphi_1 \cap \varphi_2)(a)\},$

And so $(X, \varphi_1 \cap \varphi_2)$ is not an N-sub algebra of x. For $t \in [-0.5, 0)$, we have $C(\varphi_1 \cap \varphi_2; t) = \{\theta, 1, 2, a\}$ which is not an ideal of X since $b * a = a \in C(\varphi_1 \cap \varphi_2; t)$ and $b \notin C(\varphi_1 \cap \varphi_2; t)$. Hence $(X, \varphi_1 \cap \varphi_2)$ is not an N-ideal of X.

Theorem 5.12. Let (X, φ) be an N-sub algebra (resp. N-ideal) of X. If (X, ω_1) and (X, ω_2) are retrenched N-sub algebra (resp. retrenched N-ideals) of (X, φ), then the union $(X, \omega_1 \cup \omega_2)$ is a retrenched N-sub algebra (resp. retrenched N-ideal) of (X, φ).

Proof. Clearly $(X, \omega_1 \cup \omega_2)$ is a retrenchment of (X, φ). Since $(X, \omega_1)$and (X, ω_2) are retrenched N-sub algebras (resp. N-ideals) of (X, φ) , it follows that $(X, \omega_1 \cup \omega_2)$ is an N-sub algebra (resp. N-ideal) of X. Therefore $(X, \omega_1 \cup \omega_2)$ is a retrenched N-sub algebra (resp. retrenched N-ideal) of x. Therefore $(X, \omega_1 \cup \omega_2)$ is a retrenched N-sub algebra (resp. retrenched N-ideal) of (X, φ).

Let (X, φ) be an N-sub algebra (resp. N-ideal) of X and let $\propto, \beta \in [\perp, 0]$. Then the $\propto$ −translarion $(X, {\varphi_\alpha}^T)$ and the β-translarion $(X, {\varphi_\alpha}^T)$ are N-sub algebra (resp. N-ideal) of X. If $\propto \leq \beta$, then ${\varphi_\alpha}^T(x) = \varphi(x) + \alpha \leq \varphi(x) + \beta = {\varphi_\beta}^T(x)$ for all $x \in X$, and hence $(X, {\varphi_\alpha}^T)$ is a retrenchment of $(X, {\varphi_\beta}^T)$. Therefore, we have the following theorem.

Theorem 5.13. Let (X, φ) be an N-sub algebra (resp. N-ideal) of X and let $\propto, \beta \in [\perp, 0]$. If $\propto \leq \beta$, then the $\propto$-translation $(X, {\varphi_\alpha}^T)$ of (X, φ) is a retrenched N-sub algebra (resp. retrenched N-ideal) of theβ−translation $(X, {\varphi_\beta}^T)$ of (X, φ).

For every N-sub algebra (resp. N-ideal)(X, φ) of X and $\beta \in [\perp, 0]$ the β −translation $(X, {\varphi_\beta}^T)$ is an N-sub algebra (resp. N-ideal) of X. If (X, ω) is a retrenched N-sub algebra 9resp. retrenched N-ideal) of $(X, {\varphi_\beta}^T)$, then there exists $\propto \in [\perp, 0]$ such that $\propto \leq \beta$ and $\omega(x) \leq {\varphi_\alpha}^T(x)$ for all $x \in X$.

Theorem 5.14. Let (X, φ) be an N-sub algebra (resp. N-ideal) of X and let$\beta \in [\perp, 0]$. For every retrenched n-sub algebra (resp. retrenched N-ideal) (X, ω) of the β – translation $\big(X, {\varphi_\beta}^T\big)$ of (X, φ), there exists $\propto \in [\perp, 0]$ such that $\propto \leq \beta$ and (X, ω) is a retrenched N-sub algebra (resp. retrenched N-ideal) of the $\propto$-translation $(X, {\varphi_\propto}^T)$ of (X, φ).

The following examples illustrate Theorem 5.14.

Example 5.15. Consider a BCIK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	a	a
b	b	b	θ	b	b
c	c	c	c	θ	c
d	d	d	d	d	θ

Define an N-function φ on X by

X	θ	a	b	c	d
φ	-0.7	-0.4	-0.2	-0.5	-0.1

Then (X, φ) is an N-sub algebra of X and $\perp = -0.3$. If we take $\beta = -0.15$. then the β-translation $\big(X, {\varphi_\beta}^T\big)$ of (X, φ) is given by

X	θ	a	b	c	d
${\varphi_\beta}^T$	-0.85	-0.55	-0.35	-0.65	-0.25

Let ω be an N-function on X defined by

X	θ	a	b	c	d
${\varphi_\beta}^T$	-0.89	-0.57	-0.38	-0.66	-0.28

Then (X,ω) is clearly an N-sub algebra of X which retrenchment of $\big(X, {\varphi_\beta}^T\big)$ and so (X,ω) is a retrenched N-sub algebra of the β-translation $\big(X, {\varphi_\beta}^T\big)$of (X, φ). If we take $\propto = -0.23$, then $\propto = -0.23 < -0.15 = \beta$ and the $\propto$-translation$\big(X, {\varphi_\beta}^T\big)$ of (X, φ) is given as follows:

X	θ	a	b	c	d
${\varphi_\alpha}^T$	-0.93	-0.63	-0.43	-0.73	-0.33

Note that $\omega(x) \leq {\varphi_\alpha}^T(x)$ for all $x \in X$, and hence (X,ω) is a retrenched N-sub algebra of the $\propto$-translation $\big(X, {\varphi_\beta}^T\big)$ of (X, φ).

Example 5.16. Consider a BCIK-algebra $X = (\theta, 1, a, b, c\}$ with the following Cayley table:

$*$	θ	1	a	b	c
θ	θ	θ	c	b	a
1	1	θ	c	b	a
a	a	a	θ	c	b
b	b	b	a	θ	c
c	c	c	b	a	θ

Define an N-function φ on X by

X	θ	1	a	b	c
φ	-0.65	-0.53	-0.22	-0.38	-0.22

Then (X,ω) is an N-ideal of X and $\perp = -0.35$. If we take $\beta = -0.2$ then the β-translation $\left(X, {\varphi_\beta}^T\right)$ of (X, φ) is given by

X	θ	1	a	b	c
${\varphi_\beta}^T$	-0.85	-0.73	-0.42	-0.58	-0.42

Let ω be an N-function on X defined by

X	θ	1	a	b	c
${\varphi_\beta}^T$	-0.87	-0.75	-0.45	-0.59	-0.45

Then (X,ω) is clearly an N-ideal of X which is a retrenchment of $\left(X, {\varphi_\beta}^T\right)$, and so (X,$\omega$)is a retrenched N-ideal of the β-translation $\left(X, {\varphi_\beta}^T\right)$ of (X, φ) is given as follows:

X	θ	1	a	b	c
${\varphi_\propto}^T$	-0.86	-0.74	-0.43	-0.59	-0.43

Note that $\omega(x) \leq {\varphi_\propto}^T(x)$ for all $x \in X$, and hence (X,ω) is a retrenched N-ideal of the ∝-translation $(X, {\varphi_\alpha}^T)$ of (X, φ).

6. Conclusion

In [3] see that prime ideal are irreducible in any BCIK-algebras and we verify some use full properties of ideals in BCIK-algebra such as relation between prime ideals and Maximal ideals.

In this paper introduce N-sub algebras and (commutative) N-ideals, the translations of N-subalgebras and N-ideals.

In our future study some useful properties of thisa complete ideal in extended in various algebraic structure of B-algebras, Q-algebras, subtraction algebras, d-algebra and so forth.

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