International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 5 Issue 3, March-April 2021 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

Regularity of Generalized Derivations in P-Semi Simple BCIK-Algebras

S Rethina Kumar

Assistant Professor, Department of Mathematics, Bishop Heber College
(Affiliation to Bharathidasan University), Tiruchirappalli, Tamil Nadu, India

ABSTRACT

In this paper we study the regularity of inside(or outside) $(\theta;\varphi)$-derivations in p-semi simple BCIK – algebra X and prove that let $d_{(\theta,\phi)}$: X $\rightarrow$X be an inside (θ,ϕ)-derivation of X. If there exists a$\in$ X such that $d_{(\theta,\phi)}(x) * \theta(a) = 0$, then $d_{(\theta,\phi)}$ is regular for all x$\in$ X. It is also show that if X is a BCIK-algebra, then every inside(or outside) (θ,ϕ)-derivation of X is regular. Furthermore the concepts of θ- ideal, ϕ-ideal and invariant inside (or outside) (θ,ϕ)-derivation of X are introduced and their related properties are investigated. Finally we obtain the following result: If $d_{(\theta,\phi)}$: X $\rightarrow$X is an outside (θ,ϕ)-derivation of X, then $d_{(\theta,\phi)}$ is regular if and only if every ϕ-ideal of X is $d_{(\theta,\phi)}$-invariant.

KEYWORDS: *BCIK-algebra, p-semi simple, Derivations, Regularity*

How to cite this paper: S Rethina Kumar "Regularity of Generalized Derivations in P-Semi Simple BCIK-Algebras" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-5 | Issue-3, April 2021, pp.717-727, URL: www.ijtsrd.com/papers/ijtsrd39949.pdf

IJTSRD39949

Copyright © 2021 by author(s) and International Journal of Trend in Scientific Research and Development Journal. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (CC BY 4.0) (http://creativecommons.org/licenses/by/4.0)

1. INTRODUCTION

This In 1966, Y. Imai and K. Iseki [1,2] defined BCK – algebra in this notion originated from two different sources: one of them is based on the set theory the other is form the classical and non – classical propositional calculi. In 2021 [6], S Rethina Kumar introduce combination BCK–algebra and BCI–algebra to define BCIK–algebra and its properties and also using Lattices theory to derived the some basic definitions, and they also the idea introduced a regular f-derivation in BCIK-algebras. We give the Characterizations f-derivation p-semi simple algebra and its properties. In 2021[4], S Rehina Kumar have given the notion of t-derivation of BCIK-algebras and studied p-semi simple BCIK—algebras by using the idea of regular t-derivation in BCIK-algebras have extended the results of BCIK-algebra in the same paper they defined and studied the notion of left derivation of BCIK-algebra and investigated some properties of left derivation in p-semi simple BCIK-algebras. In 2021 [7], S Rethina Kumar have defined the notion of Regular left derivation and generalized left derivation determined by a Regular left derivation on p-semi simple BCIK-algebra and discussed some related properties. Also, In 2021 [3,4,5], S Rethina Kumar have introduced the notion of generalized derivation in BCI-algebras and established some results.

The present paper X will denote a BCIK-algebra unless otherwise mentioned. In 2021[3,4,5,6,7], S Rethina Kumar defined the notion of derivation on BCIK-algebra as follows: A self-map d: X $\rightarrow$X is called a left-right derivation (briefly on (l, r)-derivation) of X if d(x * y) = d(x) * y $\wedge$ x * d(y) holds for all x, y $\in$ X. Similarly, a self-map d: X $\rightarrow$X is called a right-left derivation (briefly an (r, l)-derivation) of X if d(x * y) = x * d(y) $\wedge$ d(x) * y holds for all x, y$\in$ X. Moreover if d is both (l, r)-and (r, l)-derivation, it is a derivation on X. Following [3,4,5,6], a self-map d_f: X $\rightarrow$X is said to be a right-left f-derivation or an (l, r)-f-derivation or an (l, r)-f-derivation of X if it satisfies the identity d_f (x * y) = d_f(x) * f(y) $\wedge$ f(x) * d_f(y) for all x, y $\in$ X. Similarly, a self-map d_f: X $\rightarrow$X is said to be a right-left f-derivation or an (r, l)-f-derivation of X if it satisfies the identity d_f (x * y) = f(x) * d_f(y) $\wedge$ d_f(x) * f(y) for all x, y $\in$ X. Moreover, if d_f is an f-derivation, where f is an endomorphism. Over the past decade, a number of research papers have been devoted to the study of various kinds of derivations in BCIK-algebras (see for [3,4,5,6,7] where further references can be found).

The purpose of this paper is to study the regularity of inside (or outside) (θ,ϕ)-derivation in BCIK-algebras X and their useful properties. We prove that let $d_{(\theta,\phi)}$: X $\rightarrow$X be an inside (θ,ϕ)-derivation of X and if there exists a $\in$ X such that $d_{(\theta,\phi)}(x)(x) * \theta(a) = 0$, then $d_{(\theta,\phi)}$ is regular for all x $\in$ X. It is derivation of X is regular. Furthermore, we introduce the concepts of θ-ideal, ϕ-ideal and invariant inside (or outside) (θ,ϕ)-derivation of X and investigated their related properties. We also prove that if $d_{(\theta,\phi)}$: X $\rightarrow$X is an outside (θ,ϕ)-derivation of X, then $d_{(\theta,\phi)}$ is regular if and only if every θ-ideal of X is $d_{(\theta,\phi)}$ –invariant.

2. Preliminaries

Definition 2.1: [5] BCIK algebra

Let X be a non-empty set with a binary operation * and a constant 0. Then (X, *, 0) is called a BCIK Algebra, if it satisfies the following axioms for all x, y, z Є X:

(BCIK-1) x*y = 0, y*x = 0, z*x = 0 this imply that x = y = z.

(BCIK-2) ((x*y) * (y*z)) * (z*x) = 0.

(BCIK-3) (x*(x*y)) * y = 0.

(BCIK-4) x*x = 0, y*y = 0, z*z = 0.

(BCIK-5) 0*x =0, 0*y = 0, 0*z = 0.

For all x, y, z Є X. An inequality ≤ is a partially ordered set on X can be defined x ≤ y if and only if

(x*y) * (y*z) = 0.

Properties 2.2: [5] I any BCIK – Algebra X, the following properties hold for all x, y, z Є X:

1. 0 Є X.
2. x*0 = x.
3. x*0 = 0 implies x = 0.
4. 0*(x*y) = (0*x) * (0*y).
5. X*y = 0 implies x = y.
6. X*(0*y) = y*(0*x).
7. 0*(0*x) = x.
8. x*y Є X and x Є X imply y Є X.
9. (x*y) * z = (x*z) * y
10. x*(x*(x*y)) = x*y.
11. (x*y) *(y*z) = x*y.
12. 0 ≤ x ≤ y for all x, y Є X.
13. x ≤ y implies x*z ≤ y*z and z*y ≤ z*x.
14. x*y ≤ x.
15. x*y ≤ z ⇔ x*z ≤ y for all x, y, z Є X
16. x*a = x*b implies a = b where a and b are any natural numbers (i. e)., a, b Є N
17. a*x = b*x implies a = b.
18. a*(a*x) = x.

Definition 2.3: [4, 5, 10], Let X be a BCIK – algebra. Then, for all x, y, z Є X:

1. X is called a positive implicative BCIK – algebra if (x*y) * z = (x*z) * (y*z).
2. X is called an implicative BCIK – algebra if x*(y*x) = x.
3. X is called a commutative BCIK – algebra if x*(x*y) = y*(y*x).
4. X is called bounded BCIK – algebra, if there exists the greatest element 1 of X, and for any
5. x Є X, 1*x is denoted by GG_x,
6. X is called involutory BCIK – algebra, if for all x Є X, GG_x = x.

Definition 2.4: [5] Let X be a bounded BCIK-algebra. Then for all x, y Є X:

1. G1 = 0 and G0 = 1,
2. GG_x≤ x that GG_x = $G(G_x)$,
3. G_x * G_y≤ y*x,
4. y ≤ x implies G_x ≤ G_y,
5. $G_{x*y} = G_{y*x}$
6. $GGG_x = G_x$.

Theorem 2.5: [5] Let X be a bounded BCIK-algebra. Then for any x, y Є X, the following hold:

1. X is involutory,
2. x*y = G_y * G_x,
3. x*G_y = y * G_x,
4. x ≤ G_y implies y ≤ G_x.

Theorem 2.6: [5] Every implicative BCIK-algebra is a commutative and positive implicative BCIK-algebra.

Definition 2.7: [4,5] Let X be a BCIK-algebra. Then:

1. X is said to have bounded commutative, if for any x, y Є X, the set A(x,y) = {t Є X: t*x ≤ y} has the greatest element which is denoted by x o y,
2. (X, *, ≤) is called a BCIK-lattices, if (X, ≤) is a lattice, where ≤ is the partial BCIK-order on X, which has been introduced in Definition 2.1.

Definition 2.8: [5] Let X be a BCIK-algebra with bounded commutative. Then for all x, y, z Є X:

1. y ≤ x o (y*x),
2. (x o z) * (y o z) ≤ x*y,
3. (x*y) * z = x*(y o z),
4. If x ≤ y, then x o z ≤ y o z,
5. z*x ≤ y ⇔ z ≤ x o y.

Theorem 2.9: [4,5] Let X be a BCIK-algebra with condition bounded commutative. Then, for all x, y, z Є X, the following are equivalent:

1. X is a positive implicative,
2. x ≤ y implies x o y = y,
3. x o x = x,
4. (x o y) * z = (x*z) o (y*z),
5. x o y = x o (y*x).

Theorem 2.10: [4,5] Let X be a BCIK-algebra.

1. If X is a finite positive implicative BCIK-algebra with bounded and commutative the (X, ≤) is a distributive lattice,
2. If X is a BCIK-algebra with bounded and commutative, then X is positive implicative if and only if (X, ≤) is an upper semi lattice with x ∨ y = x o y, for any x, y Є X,
3. If X is bounded commutative BCIK-algebra, then BCIK-lattice (X, ≤) is a distributive lattice, where x ∧ y = y*(y*x) and x ∨ y= $G(G_x \wedge G_y)$.

Theorem 2.11: [4,5] Let X be an involutory BCIK-algebra, Then the following are equivalent:

1. (X, ≤) is a lower semi lattice,
2. (X, ≤) is an upper semi lattice,
3. (X, ≤) is a lattice.

Theorem 2.12: [5] Let X be a bounded BCIK-algebra. Then:

1. every commutative BCIK-algebra is an involutory BCIK-algebra.
2. Any implicative BCIK-algebra is a Boolean lattice (a complemented distributive lattice).

Theorem 2.13: [5, 11] Let X be a BCK-algebra, Then, for all x, y, z Є X, the following are equivalent:

1. X is commutative,
2. x*y = x*(y*(y*x)),
3. x*(x*y) = y*(y*(x*(x*y))),
4. x≤ y implies x = y*(y*x).

3. Regular Left derivation p-semi simple BCIK-algebra

Definition 3.1: Let X be a p-semi simple BCIK-algebra. We define addition + as x + y = x*(0*y) for all

x, y Є X. Then (X, +) be an abelian group with identity 0 and x –y = x*y. Conversely, let (X, +) be an abelian group with identity 0 and let x – y =x*y. Then X is a p-semi simple BCIK-algebra and x + y = x*(0*y),

for all x, y Є X (see [6]). We denote x ・y = y * (y * x), 0 * (0 * x) = a_x and

$L_p(X)$ = {a Є X / x * a = 0 implies x = a, for all x Є X}.

For any x Є X. V(a) = {a Є X / x * a = 0} is called the branch of X with respect to a. We have

x * y Є V (a * b), whenever x Є V(a) and y Є V(b), for all x, y Є X and all a, b Є $L_p(X)$, for 0 * (0 * a_x) = a_x which implies that a_x * y Є $L_p(X)$ for all y Є X. It is clear that G(X) ⊂ $L_p(X)$ and x * (x * a) = a and

a * x Є $L_p(X)$, for all a Є $L_p(X)$ and all x Є X.

Definition 3.2: ([5]) Let X be a BCIK-algebra. By a (l, r)-derivation of X, we mean a self d of X satisfying the identity

d(x * y) = (d(x) * y) ∧ (x * d(y)) for all x, y Є X.

If X satisfies the identity

d(x * y) = (x * d(y)) ∧ (d(x) * y) for all x, y Є X,

then we say that d is a (r, l)-derivation of X

Moreover, if d is both a (r, l)-derivation and (r, l)-derivation of X, we say that d is a derivation of X.

Definition 3.3: ([5]) A self-map d of a BCIK-algebra X is said to be regular if d (0) = 0.

Definition 3.4: ([5]) Let d be a self-map of a BCIK-algebra X. An ideal A of X is said to be d-invariant, if d(A) = A.

In this section, we define the left derivations

Definition 3.5: Let X be a BCIK-algebra By a left derivation of X, we mean a self-map D of X satisfying

D(x * y) = (x * D(y)) ∧ (y * D(x)), for all x, y Є X.

Example 3.6: Let X = {0,1,2} be a BCIK-algebra with Cayley table defined by

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a map D: X ⟶X by

$$D(x) = \begin{cases} 2 if x = 0,1 \\ 0 if x = 2. \end{cases}$$

Then it is easily checked that D is a left derivation of X.

Proposition 3.7: Let D be a left derivation of a BCIK-algebra X. Then for all x, y Є X, we have

1. x * D(x) = y * D(y).
2. D(x) = $a_{D(x)\cdot x}$.
3. D(x) = D(x) ∧ x.
4. D(x) Є $L_p(X)$.

Proof.

(1) Let x, y Є X. Then

D(0) = D(x * x) = (x * D(x)) ∧ (x * D(x)) = x * D(x).

Similarly, D(0) = y * D(y). So, D(x) = y * D(y).

2) Let x Є X. Then
D(x) = D(x * 0)
= (x * D(0)) ∧ (0 * D(x))
= (0 * D(x)) * ((0 * D(x)) * (x * D(0)))
≤ 0 * (0 * (x * D(x))))
= 0 * (0 * (x * (x * D(x))))
= 0 * (0 * (D(x) ∧ x))
= $a_{D(x)\cdot x}$.

Thus D(x) ≤ $a_{D(x)\cdot x}$. But
$a_{D(x)\cdot x}$ = 0(0 * (D(x) ∧ x)) ≤ D(x) ∧ x ≤ D(x).

Therefore, D(x) = $a_{D(x)\cdot x}$.
(1) Let x Є X. Then using (2), we have
D(x) = $a_{D(x)\cdot x}$ ≤ D(x) ∧ x .

But we know that D(x) ∧ x ≤ D(x), and hence (3) holds.

(2) Since a_x Є $L_p(X)$, for all x Є X, we get D(x) Є $L_p(X)$ by (2).

Remark 3.8: Proposition 3.3(4) implies that D(X) is a subset of $L_p(X)$.

Proposition 3.9: Let D be a left derivation of a BCIK-algebra X. Then for all x, y Є X, we have
1. Y * (y * D(x)) = D(x).
2. D(x) * y Є $L_p(X)$.

Proposition 3.10: Let D be a left derivation of a BCIK-algebra of a BCIK-algebra X. Then
1. D(0) Є $L_p(X)$.
2. D(x) = 0 + D(x), for all x Є X.
3. D(x + y) = x + D(y), for all x, y Є $L_p(X)$.
4. D(x) = x, for all x Є X if and only if D(0) = 0.
5. D(x) Є G(X), for all x Є G(X).

Proof.
1. Follows by Proposition 3.3(4).
2. Let x Є X. From Proposition 3.3(4), we get D(x) = $a_{D(x)}$, so we have
D(x) = $a_{D(x)}$ = 0 * (0 * D(x)) = 0 + D(x).

3. Let x, y Є $L_p(X)$. Then
D(x + y) = D(x * (0 * y))
= (x * D(0 * y)) ∧ ((0 * y) * D(x))
= ((0 * y) * D(x)) * (((0 * y) * D(x) * (x * D(0 * y)))
= x * D(0 * y)
= x * ((0 * D(y)) ∧ (y * D(0)))
= x * D(0 *y)
= x * (0 * D(y))
= x + D(y).

4. Let D(0) = 0 and x Є X. Then
D(x) = D(x) ∧ x = x * (x * D(x)) = x * D(0) = x * 0 = x.
Conversely, let D(x) = x, for all x Є X. So it is clear that D(0) = 0.

5. Let x Є G(x). Then 0 * = x and so
D(x) = D(0 * x)

= (0 * D(x)) ∧ (x * D(0))
= (x * D(0)) * ((x * D(0)) * (0 * D(x))
= 0 * D(x).

This give D(x) Є G(X).

Remark 3.11: Proposition 3.6(4) shows that a regular left derivation of a BCIK-algebra is the identity map. So we have the following:

Proposition 3.12: A regular left derivation of a BCIK-algebra is trivial.

Remark 3.13: Proposition 3.6(5) gives that D(x) Є G(X) ⊆ L_p(X).

Definition 3.14: An ideal A of a BCIK-algebra X is said to be D-invariant if D(A) ⊂ A.

Now, Proposition 3.8 helps to prove the following theorem.

Theorem 3.15: Let D be a left derivation of a BCIK-algebra X. Then D is regular if and only if ideal of X is D-invariant.

Proof.
Let D be a regular left derivation of a BCIK-algebra X. Then Proposition 3.8. gives that D(x) = x, for all

x Є X. Let y Є D(A), where A is an ideal of X. Then y = D(x) for some x Є A. Thus

Y * x = D(x) * x = x * x = 0 Є A.

Then y Є A and D(A) ⊂ A. Therefore, A is D-invariant.

Conversely, let every ideal of X be D-invariant. Then D({0}) ⊂ {0} and hence D(0) and D is regular.

Finally, we give a characterization of a left derivation of a p-semi simple BCIK-algebra.

Proposition 3.16: Let D be a left derivation of a p-semi-simple BCIK-algebra. Then the following hold for all x, y Є X:
1. D(x * y) = x * D(y).
2. D(x) * x = D(y) * Y.
3. D(x) * x = y * D(y).

Proof.
1. Let x, y Є X. Then
D(x * y) = (x * D(y)) ∧ ∧ (y * D(x)) = x * D(y).

2. We know that
(x * y) * (x * D(y)) ≤ D(y) * y and
(y * x) * (y * D(x)) ≤ D(x) * x.

This means that
((x * y) * (x * D(y))) * (D(y) * y) = 0, and
((y * x) * (y * D(x))) * (D(x) * x) = 0.

So
((x * y) * (x * D(y))) * (D(y) * y) = ((y * x) * (y * D(x))) * (D(x) * x). (I)

Using Proposition 3.3(1), we get,
(x * y) * D(x * y) = (y * x) * D(y * x). (II)

By (I), (II) yields
(x * y) * (x * D(y)) = (y * x) * (y * D(x)).

Since X is a p-semi simple BCIK-algebra. (I) implies that
D(x) * x = D(y) * y.

3. We have, D(0) = x * D(x). From (2), we get D(0) * 0 = D(y) * y or D(0) = D(y) * y.
So D(x) * x = y * D(y).

Theorem 3.17: In a p-semi simple BCIK-algebra X a self-map D of X is left derivation if and only if and if it is derivation.

Proof.
Assume that D is a left derivation of a BCIK-algebra X. First, we show that D is a (r, l)-derivation of X. Then
D(x * y) = x * D(y)
= (D(x) * y) * ((D(x) * Y) * (x * D(y)))
= (x * D(y)) ∧ (D(x) * y).

Now, we show that D is a (r, l)-derivation of X. Then
D(x * Y) = x * D(y)
= (x * 0) * D(y)
= (x * (D(0) * D(0)) * D(y)
= (x * ((x * D(x)) * (D(y) * y))) * D(y)
= (x * ((x * D(y)) * (D(x) * y))) * D(y)
= (x * D(y) * ((x * D(y)) * (D(x) * Y))
= (D(x) * y) ∧ (x * D(y)).

Therefore, D is a derivation of X.

Conversely, let D be a derivation of X. So it is a (r, l)-derivation of X. Then
D(x * y) = (x * D(y)) ∧ (D(x) * y)
= (D(x) * y) * ((D(x) * y) * (x * D(y)))
= x * D(y) = (y * D(x)) * ((y * D(x)) * (x * D(y)))
= (x * D(y)) ∧ (y * D(x)).

Hence, D is a left derivation of X.

4. t-Derivations in BCIK-algebra /p-Semi simple BCIK-algebra

The following definitions introduce the notion of t-derivation for a BCIK-algebra.

Definition 4.1: Let X be a BCIK-algebra. Then for t Є X, we define a self-map d_t: X ⟶ X by d_t(x) = x * t

for all x Є X.

Definition 4.2: Let X be a BCIK-algebra. Then for any t Є X, a self-map d_t: X ⟶ X is called a left-rifht t-derivation or (l,r)-t-derivation of X if it satisfies the identity d_t(x * Y) = (d_t(x) * y) ∧ (x * d_t(y)) for all x, y Є X.

Definition 4.3: Let X be a BCIK-algebra. Then for any t Є X, a self-map d_t: X ⟶ X is called a left-right t-derivation or (l, r)-t-derivation of X if it satisfies the identity d_t(x * y) = (x * d_t(y)) ∧ (d_t(x) * y) for all x, y Є X.

Moreover, if d_t is both a (l, r)and a (r, l)-t-derivation on X, we say that $d_t$is a t-derivation on X.

Example 4.4: Let X = {0,1,2} be a BCIK-algebra with the following Cayley table:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

For any t Є X, define a self-map d_t: X ⟶ X by d_t(x) = x * t for all x Є X. Then it is easily checked that d_t is a t-derivation of X.

Proposition 4.5: Let d_t be a self-map of an associative BCIK-algebra X. Then d_t is a (l, r)-t-derivation of X.

Proof. Let X be an associative BCIK-algebra, then we have
d_t(x * y) = (x * y)
= {x * (y * t)} * 0
= {x * (y * t)} * [{x * (y * t)} *{x * (y *t)}]
= {x * (y * t)} * [{x * (y *t)} * {(x * y) * t}]

$= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * t) * y\}]$
$= ((x * t) * y) \wedge (x * (y * t))$
$= (d_t(x) * y) \wedge (x * d_t(y))$.

Proposition 4.6: Let d_t be a self-map of an associative BCIK-algebra X. Then, d_t is a (r, l)-t-derivation of X.

Proof. Let X be an associative BCIK-algebra, then we have
$d_t(x * y) = (x * y) * t$
$= \{(x * t) * y\} * 0$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * t) * y)]$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * y) * t\}]$
$= \{(x * t) * y\} * [\{(x * t) * y\} * \{x * (y * t)\}]$
$= (x * (y * t)) \wedge ((x * t) * y)$
$= (x * d_t(y)) \wedge (d_t(x) * y)$

Combining Propositions 4.5 and 4.6, we get the following Theorem.

Theorem 4.7: Let d_t be a self-map of an associative BCIK-algebra X. Then, d_t is a t-derivation of x.

Definition 4.8: A self-map d_t of a BCIK-algebra X is said to be t-regular if $d_t(0) = 0$.

Example 4.9: Let X = {0, a, b} be a BCIK-algebra with the following Cayley table:

*	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

1. For any t Є X, define a self-map d_t: X $\rightarrow$ X by

$$d_t(x) = x * t = \begin{cases} b \text{ if } x = 0, a \\ 0 \text{ if } x = b \end{cases}$$

Then it is easily checked that d_t is (l, r) and (r, l)-t-derivations of X, which is not t-regular.

2. For any t Є X, define a self-map d't: X $\rightarrow$ X by

$d_t'(x) = x * t = 0$ if x = 0, a, b if x = b.

Then it is easily checked that d_t' is (l, r) and (r, l)-t-derivations of X, which is t-regular.

Proposition 4.10: Let d_t be a self-map of a BCIK-algebra X. Then
1. If d_t is a (l, r)-t- derivation of x, then $d_t(x) = d_t(x) \wedge x$ for all x Є X.
2. If d_t is a (r, l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all x Є X if and only if d_t is t-regular.

Proof.
1. Let d_t be a (l, r)-t-derivation of X, then
$d_t(x) = d_t(x * 0)$
$= (d_t(x) * 0) \wedge (x * d_t(0))$
$= d_t(x) \wedge (x * d_t(0))$
$= \{x * d_t(0)\} * [\{x * d_t(0)\} * d_t(x)]$
$= \{x * d_t(0)\} * [\{x * d_t(x)\} * d_t(0)]$
$\leq x * \{x * d_t(x)\}$
$= d_t(x) \wedge x$.

But $d_t(x) \wedge x \leq d_t(x)$ is trivial so (1) holds.

2. Let d_t be a (r, l)-t-derivation of X. If $d_t(x) = x \leq d_t(x)$ then
$d_t(0) = 0 \wedge d_t(0)$
$= d_t(0) * \{ d_t(0) * 0\}$
$= d_t(0) * d_t(0)$
$= 0$

Thereby implying d_t is t-regular. Conversely, suppose that d_t is t-regular, that is $d_t(0) = 0$, then we have
$d_t(0) = d_t(x * 0)$
$= (x * d_t(0)) \wedge (d_t(x) * 0)$
$= (x * 0) \wedge d_t(x)$
$= x \wedge d_t(x)$.

The completes the proof.

Theorem 4.11: Let d_t be a (l, r)-t-derivation of a p-semi simple BCIK-algebra X. Then the following hold:
1. $d_t(0) = d_t(x) * x$ for all x Є X.
2. d_t is one-0ne.
3. If there is an element x Є X such that $d_t(x) = x$, then d_t is identity map.
4. If $x \leq y$, then $d_t(x) \leq d_t(y)$ for all x, y Є X.

Proof.
1. Let d_t be a (l, r)-t-derivation of a p-semi simple BCIK-algebra X. Then for all x Є X, we have
$x * x = 0$ and so
$d_t(0) = d_t(x * x)$
$= (d_t(x) * x) \wedge (x * d_t(x))$
$= \{x * d_t(x)\} * [\{x * d_t(x)\} * \{d_t(x) * x\}]$
$= d_t(x) * x$
2. Let $d_t(x) = d_t(y) \Rightarrow x * t = y * t$, then we have x = y and so d_t is one-one.
3. Let d_t be t-regular and x Є X. Then, $0 = d_t(0)$ so by the above part(1), we have $0 = d_t(x) * x$ and, we obtain $d_t(x) = x$ for all x Є X. Therefore, d_t is the identity map.
4. It is trivial and follows from the above part (3).
Let $x \leq y$ implying $x * y = 0$. Now,
$d_t(x) * d_t(y) = (x * t) * (y * t)$
$= x * y$
$= 0$.

Therefore, $d_t(x) \leq d_t(y)$. This completes proof.

Definition 4.12: Let d_t be a t-derivation of a BCIK-algebra X. Then, d_t is said to be an isotone t-derivation if $x \leq y \Rightarrow d_t(x) \leq d_t(y)$ for all x, y Є X.

Example 4.13: In Example 4.9(2), d_t' is an isotone t-derivation, while in Example 4.9(1), d_t is not an isotone t-derivation.

Proposition 4.14: Let X be a BCIK-algebra and d_t be a t-derivation on X. Then for all x, y Є X, the following hold:
1. If $d_t(x \wedge y) = d_t(x)\ d_t(x)\ d_t(x)$, then d_t is an isotone t-derivation
2. If $d_t(x \wedge y) = d_t(x) * d_t(y)$, then d_t is an isotone t-derivation.

Proof.
1. Let $d_t(x \wedge y) = d_t(x) \wedge d_t(x)$. If $x \leq y \Rightarrow x \wedge y = x$ for all x, y Є X. Therefore, we have
$d_t(x) = d_t(x \wedge y)$
$= d_t(x) \wedge d_t(y)$
$\leq d_t(y)$.

Henceforth $d_t(x) \leq d_t(y)$ which implies that d_t is an isotone t-derivation.

Let $d_t(x * y) = d_t(x) * d_t(y)$. If $x \leq y \Rightarrow x * y = 0$ for all x, y Є X. Therefore, we have
$d_t(x) = d_t(x * 0)$
$= d_t\{x * (x * y)\}$

$= d_t(x) * d_t(x * y)$
$= d_t(x) * \{ d_t(x) * d_t(y)\}$
$\leq d_t(y)$.

Thus, $d_t(x) \leq d_t(y)$. This completes the proof.

Theorem 4.15: Let d_t be a t-regular (r, l)-t-derivation of a BCIK-algebra X. Then, the following hold:
1. $d_t(x) \leq x$ for all x Є X.
2. $d_t(x) * y \leq x * d_t(y)$ for all x, y Є X.
3. $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$ for all x, y Є X.
4. $Ker(d_t) = \{ x Є X: d_t(x) = 0\}$ is a sub algebra of X.

Proof.
1. For any x Є X,

we have $d_t(x) = d_t(x * 0) = (x * d_t(0)) \wedge (d_t(x) * 0) = (x * 0) \wedge (d_t(x) * 0) = x \wedge d_t(x) \leq x$.

2. Since $d_t(x) \leq x$ for all x Є X, then $d_t(x) * y \leq x * y \leq x * d_t(y)$ and hence the proof follows.

3. For any x, y Є X, we have

$d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$
$= \{d_t(x) * y\} * [\{d_t(x) * y\} * \{x * d_t(x)\}]$
$= \{d_t(x) * y\} * 0$
$= d_t(x) * y \leq d_t(x) * d_t(x)$.

4. Let x, y Є ker $(d_t) \Rightarrow d_t(x) = 0 = d_t(y)$. From (3), we have $d_t(x * y) \leq d_t(x) * d_t(y) = 0 * 0 = 0$ implying $d_t(x * y) \leq 0$ and so $d_t(x * y) = 0$. Therefore, x * y Є ker (d_t). Consequently, ker(d_t) is a sub algebra of X. This completes the proof.

Definition 4.16: Let X be a BCIK-algebra and let d_t,d_t' be two self-maps of X. Then we define

d_t o d_t': X ⟶ X by $(d_t \text{ o } d_t')(x) = d_t(d_t'(x))$ for all x Є X.

Example 4.17: Let X = {0, a, b} be a BCIK-algebra which is given in Example 4.4. Let d_t and d_t' be two

self-maps on X as define in Example 4.9(1) and Example 4.9(2), respectively.

Now, define a self-map d_t o d_t': X ⟶ X by

$$(d_t \text{ o } d_t')(x) = \begin{cases} 0 \, if \;\; x = a, b \\ b \, if \;\; x = 0. \end{cases}$$

Then, it easily checked that $(d_t \text{ o } d_t')(x) = d_t(d_t'(x))$ for all x Є X.

Proposition 4.18: Let X be a p-semi simple BCIK-algebra X and let d_t, d_t' be (l, r)-t-derivations of X.

Then, d_t o d_t' is also a (l, r)-t-derivation of X.

Proof. Let X be a p-semi simple BCIK-algebra. d_t and d_t' are (l, r)-t-derivations of X. Then for all x, y Є X, we get

$(d_t \text{ o } d_t')(x * y) = d_t(d_t'(x,y))$
$= d_t[(d_t'(x) * y) \wedge (x * d_t(y))]$
$= d_t[(x * d_t'(y)) * \{(x * d_t(y)) * (d_t'(x) * y)\}]$
$= d_t(d_t'(x) * y)$
$= \{x * d_t(d_t'(y))\} * [\{x * d_t(d_t'(y))\} * \{d_t(d_t'(x) * y)\}]$
$= \{ d_t(d_t'(x) * y)\} \wedge \{x * d_t(d_t'(y))\}$
$= ((d_t \text{ o } d_t')(x) * y) \wedge (x * (d_t \text{ o } d_t')(y))$.

Therefore, $(d_t \text{ o } d_t')$ is a (l, r)-t-derivation of X.

Similarly, we can prove the following.

Proposition 4.19: Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be (r, l)-t-derivations of X. Then, d_t o d_t' is also a (r, l)-t-derivation of X.

Combining Propositions 3.18 and 3.19, we get the following.

Theorem 4.20: Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be t-derivations of X. Then, d_t o d_t' is also a t-derivation of X.

Now, we prove the following theorem

Theorem 4.21: Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be t-derivations of X.
Then d_t o d_t' = d_t' o d_t.

Proof. Let X be a p-semi simple BCIK-algebra. d_t and d_t', t-derivations of X. Suppose d_t' is a
(l, r)-t-derivation, then for all x, y Є X, we have

$(d_t \text{ o } d_t')(x * y) = d_t(d_t'(x * y))$
$= d_t[(d_t'(x) * y) \wedge (x * d_t(y))]$
$= d_t[(x * d_t'(y)) * \{(x * d_t(y)) * (d_t'(x) * y)\}]$
$= d_t(d_t'(x) * y)$

As d_t is a (r, l)-t-derivation, then
$= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y)$
$= d_t'(x) * d_t(y)$.

Again, if d_t is a (r, l)-t-derivation, then we have
$(d_t \text{ o } d_t')(x * y) = d_t'[d_t(x * y)]$
$= d_t'[(x * d_t(y)) \wedge (d_t(x) * y)]$
$= d_t'[x * d_t(y)]$

But d_t' is a (l, r)-t-derivation, then
$= (d_t'(x) * d_t(y)) \wedge (x * d_t'(d_t(y))$
$= d_t'(x) * d_t(y)$

Therefore, we obtain
$(d_t \text{ o } d_t')(x * y) = (d_t' \text{ o } d_t)(x * y)$.

By putting y = 0, we get
$(d_t \text{ o } d_t')(x) = (d_t' \text{ o } d_t)(x)$ for all x Є X.

Hence, d_t o d_t' = d_t' o d_t. This completes the proof.

Definition 4.22: Let X be a BCIK-algebra and let d_t,d_t' two self-maps of X. Then we define $d_t * d_t'$: X ⟶ X by $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all x Є X.

Example 4.23: Let X = {0, a, b} be a BCIK-algebra which is given in Example 3.4. let d_t and d_t' be two

Self-maps on X as defined in Example 4.9 (1) and Example 4.10 (2), respectively.

Now, define a self-map $d_t * d_t'$: X ⟶ X by $(d_t * d_t')(x) =$

$$\begin{cases} 0 \, if \; x = a, b \\ b \, if \; x = 0. \end{cases}$$

Then, it is easily checked that $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all x Є X.

Theorem 4.24: Let X be a p-semi simple BCIK-algebra and let d_t,d_t' be t-derivations of X.

Then $d_t * d_t' = d_t' * d_t$.

Proof. Let X be a p-semi simple BCIK-algebra. d_t and d_t', t-derivations of X.

Since d_t' is a (r, l)-t-derivation of X, then for all x, y Є X, we have

$(d_t \circ d_t')(x * y) = d_t(d_t'(x * y))$
$= d_t[(x * d_t'(y)) \wedge (d_t'(x) * y)]$
$= d_t[(x * d_t'(y)]$

But $d_t$is a (l, r)-r-derivation, so
$= (d_t(x) * d_t'(y)) \wedge (x * d_t(d_t'(y))$
$= d_t(x) * d_t'(x)$.

Again, if d_t' is a (l, r)-t-derivation of X, then for all x, y Є X, we have
$(d_t \circ d_t')(x * y) = d_t[d_t'(x * y)]$
$= d_t[(d_t'(x) * y) \wedge (x * d_t'(y))]$
$= d_t[(x * d_t'(y)) * \{(x * d_t'(y)) * (d_t'(x) * y)\}]$
$= d_t(d_t'(x) * y)$.

As d_t is a (r, l)-t-derivation, then
$= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y)$
$= d_t'(x) * d_t(y)$.

Henceforth, we conclude
$d_t(x) * d_t'(y) = d_t'(x) * d_t(y)$
By putting y =x, we get
$d_t(x) * d_t'(x) = d_t'(x) * d_t(x)$
$(d_t * d_t')(x) = (d_t' * d_t)(x)$ for all x Є X.
Hence$d_t * d_t' = d_t' * d_t$. This completes the proof.

5. f-derivation of BCIK-algebra

In what follows, let be an endomorphism of X unless otherwise specified.

Definition 5.1: Let X be a BCIK algebra. By a left f-derivation (briefly, (l, r)-f-derivation) of X, a self-map $d_f(x * y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$ for all x, y Є X is meant, where f is an endomorphism of X. If d_f satisfies the identity $d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$ for all x, y Є X, then it is said that d_f is a right-left f-derivation (briefly, (r, l)-f-derivation) of X. Moreover, if d_f is both an (r, l)-f-derivation, it is said that d_f is an f-derivation.

Example 5.2: Let X = {0,1,2,3,4,5} be a BCIK-algebra with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Define a Map d_f: X $\longrightarrow$ X by

$$d_f = \begin{cases} 2 \, if \;\; x = 0,1, \\ 0 \, otherwise, \end{cases}$$

and define and endomorphism f of X by

$$f(x) = \begin{cases} 2 \, if \;\; x = 0,1, \\ 0 \, otherwise, \end{cases}$$

That it is easily checked that d_f is both derivation and f-derivation of X.

Example 5.3: Let X be a BCIK-algebra as in Example 2.2. Define a map d_f: X $\longrightarrow$ X by

$$d_f = \begin{cases} 2 \, if \;\; x = 0,1, \\ 0 \, otherwise, \end{cases}$$

Then it is easily checked that d_f is a derivation of X.

Define an endomorphism f of X by

f(x) = 0, for all x Є X.

Then d_f is not an f-derivation of X since

$d_f(2 * 3) = d_f(0) = 2$,

but

$(d_f(2) * f(3)) \wedge (f(2) * d_f(3)) = (0*0) \wedge (0*0) = 0 \wedge 0 = 0$,

And thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \wedge (f(2) * d_f(3))$.

Remark 5.4: From Example 5.3, we know that there is a derivation of X which is not an f-derivation X.

Example 2.5: Let X = {0,1,2,3,4,5} be a BCIK-algebra with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	1	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Define a map d_f: X $\longrightarrow$ X by

$$d_f(x) = \begin{cases} 0 \; if \;\; x = 0,1, \\ 2 \; if \;\; x = 2,4, \\ 3 \; if \;\; x = 3,5, \end{cases}$$

and define an endomorphism f of X by

$$f(x) = \begin{cases} 0 \; if \;\; x = 0,1, \\ 2 \; if \;\; x = 2,4, \\ 3 \; if \;\; x = 3,5, \end{cases}$$

Then it is easily checked that d_f is both derivation and f-derivation of X.

Example 5.6: Let X be a BCIK-algebra as in Example 5.5. Define a map d_f: X $\longrightarrow$ X by

$$d_f(x) = \begin{cases} 0 \; if \;\; x = 0,1, \\ 2 \; if \;\; x = 2,4, \\ 3 \; if \;\; x = 3,5, \end{cases}$$

Then it is easily checked that d_f is a derivation of X.

Define an endomorphism f of X by

f (0) = 0, f (1) = 1, f(2) = 3 f (3) = 2, f (4) = 5, f (5) = 4.

Then d_f is not an f-derivation of X since

$d_f(2 * 3) = d_f(3) = 3$,

but

$(d_f(2) * f(3)) \wedge (f(2) * d_f(3)) = (2*2) \wedge (3*3) = 0 \wedge 0 = 0$,

And thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \wedge (f(2) * d_f(3))$.

Example 5.7: Let X be a BCIK-algebra as in Example 2.5. Define a map d_f: X $\longrightarrow$ X by$d_f(0) = 0$, $d_f(1) = 1$,$d_f(2) = 3$,$d_f(3) = 2$, $d_f(4) = 5$, $d_f(5) = 4$, Then d_f is not a derivation of X since

$d_f(2 * 3) = d_f(3) = 2$,

$(d_f(2) * 3) \wedge (2 * d_f(3)) = (3 * 3) \wedge (2 * 2) = 0 \wedge 0 = 0$,

And thus And thus $d_f(2 * 3) \neq (d_f(2) * 3) \wedge (2 * d_f(3))$.

Define an endomorphism f of X by

$f(0) = 0, f(1) = 1, f(2) = 3, f(3) = 2, f(4) = 5, f(5) = 4$.

Then it is easily checked that d_f is an f-derivation of X.

Remark 5.8: From Example 5.7, we know there is an f-derivation of X which is not a derivation of X.

For convenience, we denote $f_x = 0 * (0 * f(x))$ for all $x \in X$. Note that $f_x \in L_p(X)$.

Theorem 5.9: Let d_f be a self-map of a BCIK-algebra X define by $d_f(x) = f_x$ for all $x \in X$.

Then d_f is an (l, r)-f-derivation of X. Moreover, if X is commutative, then d_f is an (r, l)-f-derivation of X.

Proof. Let $x, y \in X$

Since
$0 * (0 * (f_x * f(y))) = 0 * (0 * ((0 * (0 * f(x)) * f(y)))$
$= 0 * ((0 * ((0 * f(y)) * (0 * f(x))))$
$= 0 * (0 * (0 * f(y * x))) = 0 * f(y * x)$
$= 0 * (f(y) * f(x)) = (0 * f(y)) * (0 * f(x))$
$= (0 * (0 * f(x))) * f(y) = f_x * f(y)$,

We have $f_x * f(y) \in L_p(X)$, and thus
$f_x * f(y) = (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y)))$,

It follows that
$d_f(x * x) = f_{x*x} = 0 * (0 * f(x*y)) = 0 * (0 * (f(x) * f(y)))$
$= (0 * (0 * f(x)) * (0 * (0 * f(y))) = f_x * f_y$
$= (0 * (0 * f_x)) * (0 * (0 * f(y))) = 0 * (0 * (f_x * f(y)))$
$= f_x * f(y) = (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y)))$
$= (f_x * f(y)) \wedge (f(x) \wedge f_y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$,

And so d_f is an (l, r)-f-derivation of X. Now, assume that X is commutative. So $d_f(x) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch $x, y \in X$, we have
$d_f(x) * f(y) = f_x * f(y) = (0 * (f_x * f(y)))$
$= (0 * (0 * f_x)) * (0 * (0 * f(y)))$
$= f_x * f_x \in V(f_x * f_x)$,

And so $f_x * f_x = (0 * (0 * f(x))) * (0 * (0 * f_y)) = 0 * (0 * (f(x) * f_y)) = 0 * (0 * (f(x) * d_f(y)) \leq f(x) * d_f(y)$, which implies that $f(x) * d_f(y) \in V(f_x * f_x)$. Hence, $d_f(y) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch, and so

$d_f(x * x) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$
$= (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$.

This completes the proof.

Proposition 5.10: Let d_f be a self-map of a BCIK-algebra. Then the following hold.

1. If d_f is an (l, r)-f-derivation of X, then $d_f(x) = d_f(x) \wedge f(x)$ for all $x \in X$.
2. If d_f is an (r, l)-f-derivation of X, then $d_f(x) = f(x) \wedge d_f(x)$ for all $x \in X$ if and only if $d_f(0) = 0$.

Proof.
1. Let d_f is an (r, l)-f-derivation of X, Then,
$d_f(x) = d_f(x * 0) = (d_f(x) * f(0)) \wedge (f(x) * d_f(0))$
$= (d_f(x) * 0) \wedge (f(x) * d_f(0)) = d_f(x) \wedge (f(x) * d_f(0))$
$= (f(x) * d_f(0)) * ((f(x) * d_f(0)) * d_f(x))$
$= (f(x) * d_f(0)) * ((f(x) * d_f(0)) * d_f(0))$
$\leq f(x) * (f(x) * d_f(x)) = d_f(x) \wedge f(x)$.

But $d_f(x) \wedge f(x) \leq d_f(x)$ is trivial and so (1) holds.

2. Let d_f be an (r, l)-f-derivation of X. If $d_f(x) = f(x) * d_f(x)$ for all $x \in X$, then for $x = 0$, $d_f(0) = f(0) * d_f(0) = 0 \wedge f(0) = d_f(0) * (d_f(0) * 0) = 0$.

Conversely, if $d_f(0) = 0$, then $d_f(x) = d_f(x * 0) = (f(x) * (d_f(0)) \wedge (d_f(x) * f(0)) =$

$(f(x) * 0)) \wedge (d_f(x) * 0) = f(x) \wedge d_f(x)$, ending the proof.

Proposition 5.11: Let d_f be an (l, r)-f-derivation of a BCIK-algebra X. Then,

1. $d_f(x) \in L_p(X)$, then is $d_f(0) = 0 * (0 * d_f(x))$;
2. $d_f(a) = d_f(0) * (0 * f(a)) = d_f(0) + f(a)$ for all $a \in L_p(X)$;
3. $d_f(a) \in L_p(X)$ for all $a \in L_p(X)$;
4. $d_f(a + b) = d_f(a) + d_f(b) - d_f(0)$ for all $a, b \in L_p(X)$.

Proof.
1. The proof follows from Proposition 5.10(1).
2. Let $a \in L_p(X)$, then $a = 0 * (0 * a)$, and so $f(a) = 0 * (0 * f(a))$, that is, $f(b) \in L_p(X)$.

Hence
$d_f(a) = d_f(0 * (0 * a))$
$= (d_f(0) * f(0 * a)) \wedge (f(0) * d_f(0 * a))$
$= (d_f(0) * f(0 * a)) \wedge (0 * d_f(0 * a))$
$= (0 * d_f(0 * a)) * ((0 * d_f(0 * a)) * (d_f(0) * f(0 * a)))$
$= (0 * d_f(0 * a)) * ((0 * (d_f(0) * f(0 * a))) * d_f(0 * a))$
$= 0 * (0 * (d_f(0) * (0 * f(a))))$
$= d_f(0) * (0 * f(a)) = d_f(0) + f(a)$.

3. The proof follows directly from (2).
4. Let $a, b \in L_p(X)$. Note that $a + b \in L_p(X)$, so from (2), we note that

$d_f(a + b) = d_f(0) + f(a) + d_f(0) + f(b) - d_f(0) = d_f(a) + d_f(0) - d_f(0)$.

Proposition 5.12: Let d_f be a (r, l)-f-derivation of a BCIK-algebra X. Then,

1. $d_f(a) \in G(X)$ for all $a \in G(X)$;
2. $d_f(a) \in L_p(X)$ for all $a \in G(X)$;
3. $d_f(a) = f(a) * d_f(0) = f(a) + d_f(a)$ for all $a, b \in L_p(X)$;
4. $d_f(a + b) = d_f(a) + d_f(b) - d_f(0)$ for all $a, b \in L_p(X)$.

Proof.
1. For any $a \in G(X)$, we have $d_f(a) = d_f(0 * a) = (f(0) * d_f(a)) \wedge (d_f(0) + f(a))$
$= (d_f(0) + f(a)) * ((d_f(0) + f(a)) * (0 * d_f(0))) = 0 * d_f(0)$, and so $d_f(a) \in G(X)$.

2. For any $a \in L_p(X)$, we get
$d_f(a) = d_f(0 * (0 * a)) = (0 * d_f(0 * a)) \wedge (d_f(0) * f(0 * a))$
$= (d_f(0) * f(0 * a)) * ((d_f(0) * f(0 * a)) * (0 * d_f(0 * a)))$
$= 0 * d_f(0 * a) \in L_p(X)$.

3. For any $a \in L_p(X)$, we get
$d_f(a) = d_f(a * 0) = (f(a) * d_f(0)) \wedge (d_f(a) * f(0))$
$= d_f(a) * (d_f(a) * (f(a) * d_f(0))) = f(a) * d_f(0)$
$= f(a) * (o * d_f(0)) = f(a) + d_f(a)$.

4. The proof from (3). This completes the proof.

Using Proposition 5.12, we know there is an (l,r)-f-derivation which is not an (r,l)-f-derivation as shown in the following example.

Example 5.13: Let Z be the set of all integers and "-" the minus operation on Z. Then (Z, -, 0) is a BCIK-algebra. Let d_f: X $\rightarrow$ X be defined by $d_f(x) = f(x) - 1$ for all x Є Z.

Then, $(d_f(x) - f(y)) \wedge (f(x) - d_f(y)) = (f(x) - 1 - f(y)) \wedge (f(x) - (f(y) - 1))$
$= (f(x - Y) - 1) \wedge (f(x - y) + 1)$
$= (f(x - Y) + 1) - 2 = f(x - Y) - 1$
$= d_f(x - y)$.

Hence, d_f is an (l, r)-f-derivation of X. But $d_f(0) = f(0) - 1 = -1 \neq 1 = f(0) - d_f(0) = 0 - d_f(0)$,

that is, $d_f(0) \notin G(X)$. Therefore, d_f is not an (r, l)-f-derivation of X by Proposition 2.12(1).

6. Regular f-derivations

Definition 6.1: An f-derivation d_f of a BCIK-algebra X is said to be a regular if $d_f(0) = 0$

Remark 6.2: we know that the f-derivations d_f in Example 5.5 and 5.7 are regular.

Proposition 6.3: Let X be a commutative BCIK-algebra and let d_f be a regular (r, l)-f-derivation of X. Then the following hold.
1. Both f(x) and $d_f(x)$ belong to the same branch for all x Є X.
2. d_f is an (l, r)-f-derivation of X.

Proof.
1. Let x Є X. Then,
$0 = d_f(0) = d_f(a_x * x)$
$= (f(a_x) * d_f(x)) \wedge (d_f(a_x) * f(x))$
$= (d_f(a_x) * f(x)) * ((d_f(a_x) * f(x)) * (f(x) * d_f(a_x)))$
$= (d_f(a_x) * f(x)) * ((d_f(a_x) * f(x)) * (f(x) * d_f(a_x)))$
$= f_x * d_f(a_x)$ since $f_x * d_f(a_x) \in L_p(X)$,

And so $f_x \leq d_f(x)$. This shows that $d_f(x) \in V(X)$, Clearly, $f(x) \in V(X)$.

2. By (1), we have $f(x) * d_f(y) \in V(f_x * f_y)$ and $d_f(x) * f(y) \in V(f_x * f_y)$. Thus

$d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y)) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$, which implies that

d_f is an (l, r)-f-derivation of X.

Remark 6.4: The f-derivations d_f in Examples 5.5 and 5.7 are regular f-derivations but we know that the (l, r)-f-derivation d_f in Example 5.2 is not regular. In the following, we give some properties of regular f-derivations.

Definition 6.5: Let X be a BCIK-algebra. Then define $kerd_f = \{x \in X \,/\, d_f(x) = 0$ for all f-derivations $d_f\}$.

Proposition 6.6: Let d_f be an f-derivation of a BCIK-algebra X. Then the following hold:
1. $d_f(x) \leq f(x)$ for all x Є X;
2. $d_f(x) * f(y) \leq f(x) * d_f(y)$ for all x, y Є X;
3. $d_f(x * y) = d_f(x) * f(y) \leq d_f(x) * d_f(y)$ for all x, y Є X;
4. $kerd_f$ is a sub algebra of X. Especially, if f is monic, then $kerd_f \subseteq X_+$.

Proof.
1. The proof follows by Proposition 5.10(2).
2. Since $d_f(x) \leq f(x)$ for all x Є X, then $d_f(x) * f(y) \leq f(x) * f(y) \leq f(x) * d_f(y)$.
3. For any x, y Є X, we have

$d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$
$= (d_f(x) * f(y)) * ((d_f(x) * f(y)) * f(x) * d_f(y)))$
$= (d_f(x) * f(y)) * 0 = d_f(x) * f(y) \leq d_f(x) * d_f(y)$,

Which proves (3).

4. Let x, y Є $kerd_f$, then $d_f(x) = 0 = d_f(y)$, and so $d_f(x * y) \leq d_f(x) * d_f(y) = 0 * 0 = 0$ by (3),

and thus $d_f(x * y) = 0$, that is, x * y Є $kerd_f$, then $0 = d_f(x) \leq f(x)$ by (1), and so $f(x) \in X_+$,

that is, $0 * f(x) = 0$, and thus $f(0 * x) = f(x)$, which that $0 * x = x$, and so $x \in X_+$, that is,

$kerd_f \subseteq X_+$.

Theorem 6.7: Let be monic of a commutative BCIK-algebra X. Then X is p-semi simple if and only if

$kerd_f = \{0\}$ for every regular f-derivation d_f of X.

Proof.
Assume that X is p-semi simple BCIK-algebra and let d_f be a regular f-derivation of X. Then $X_+ = \{0\}$, and

So $kerd_f = \{0\}$ by using Proposition 6.6(4), Conversely, let $kerd_f = \{0\}$ for every regular f-derivation d_f of X. Define a self-map d_f of X by $d^*_f(0) = f_x$ for all x Є X. Using Theorem 5.9, d^*_f is an f-derivation of X. Clearly, $d^*_f(0) = f_0 = 0 * (0 * f(0)) = 0$, and so d^*_f is a regular f-derivation of X. It follows from the hypothesis that ker $d^*_f = \{0\}$. In addition, $d^*_f(x) = f_x = 0 * (0 * f(x)) = f(0 * (0 * x)) = f(0) = 0$ for all $x \in X_+$, and thus x Є ker d^*_f. Hence, by Proposition 6.6(4), $X_+ \in$ ker $d^*_f = \{0\}$. Therefore, X is p-semi simple.

Definition 6.8: An ideal A of a BCIK-algebra X is said to be an f-ideal if $f(A) \subseteq A$.

Definition 6.9: Let d_f be a self-map of a BCIK-algebra X. An f-ideal A of X is said to be d_f –invariant if

$d_f(a) \subseteq A$.

Theorem 6.10: Let d_f be a regular (r, l)-f-derivation of a BCIK-algebra X, then every f-ideal A of X is

$d_f(A) \subseteq A$.

Theorem 6.11: Let d_f be a regular (r, l)-f-derivation of a BCIK-algebra X, then every f-ideal A of X is

d_f –invariant.

Proof.
By Proposition 6.10(2), we have $d_f(x) = f(x) \wedge d_f(x) \leq f(x)$ for all x Є X. Let y Є $d_f(A)$. Let y Є $d_f(A)$.

Then $y = d_f(x)$ for some x Є A. It follows that $y * f(x) = d_f(x) * f(x) = 0 \in A$. Since x Є A, then

$f(x) \in f(A) \subseteq A$ as A is an f-ideal. It follows that y Є A since A is an ideal of X. Hence $d_f(A) \subseteq A$,

and thus A is d_f – invariant.

Theorem 6.12: Let d_f be an f-derivation of a BCIK-algebra X. Then d_f is regular if and only if every f-ideal of X is d_f-invariant.

Proof. Let d_f be a derivation of a BCIK-algebra X and assume that every f-ideal of X is d_f–invariant. Then

Since the zero ideal {0} is f-ideal and d_f –invariant, we have $d_f(\{0\}) \subseteq \{0\}$, which implies that $d_f(0) = 0$.

Thus d_f is regular. Combining this and Theorem 6.10, we complete the proof.

7. Regularity of generalized derivations

To develop our main results, the following:

Definition 7.1: [8]. Let θ *and* ϕ be two endomorphisms of X. A self-map $d_{(\theta,\phi)}$: X $\longrightarrow$ X is called

1. An inside (θ,ϕ)-derivation of $(\forall\, x,y \in X)(d_{(\theta,\phi)}(x*y) = (d_{(\theta,\phi)}(x)*\theta(y)) \wedge (\phi(x)* d_{(\theta,\phi)}(y)))$,
2. An outside (θ,ϕ)-derivation of X if it satisfies: $(\forall\, x, y \in X)\ (d_{(\theta,\phi)}(x*y) = ((\phi(x) * d_{(\theta,\phi)}(y)) \wedge (d_{(\theta,\phi)}(x) * \theta(y))$,
3. A (θ,ϕ)-derivation of X if it is both inside (θ,ϕ)-derivation and an outside (θ,ϕ)-derivation .

Example 7.2: [8]. Consider a BCIK- algebra X= {0,a,b} with the following Cayley table:

*	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Define a map

$$d_{(\theta,\phi)}: X \longrightarrow X,\ x \mapsto \begin{cases} b \text{ if } x \in \{0, a\}, \\ 0 \text{ if } x = b, \end{cases}$$

and define two endomorphisms

$$\theta: X \longrightarrow X,\ x \mapsto \begin{cases} 0 \text{ if } x \in \{0, a\}, \\ b \text{ if } x = b, \end{cases}$$

And ϕ : X $\longrightarrow$ X such that $\theta(x) = x$ for all $x \in X$.

It is routine to verify that $d_{(\theta,\phi)}$ is both an inside (θ,ϕ)-derivation and an outside (θ,ϕ)-derivation of X.

Lemma 7.3: [8]. For any outside (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCIK-algebra X, the following are equivalent:

1. $(\forall\, x \in X)\ (d_{(\theta,\phi)}(x) = \theta(x) \wedge d_{(\theta,\phi)}(x))$
2. $d_{(\theta,\phi)}(0) = 0$.

Definition 7.4: Let $d_{(\theta,\phi)}$: X $\longrightarrow$ X be an inside (or out side) (θ,ϕ)-derivation of a BCIK-algebra X. Then $d_{(\theta,\phi)}$ is said to be regular if $d_{(\theta,\phi)}(0) = 0$.

Example 7.5: The inside (or outside) (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of X in Example 7.2. is not regular.

Proposition 7.6: Let $d_{(\theta,\phi)}$ be a regular outside (θ,ϕ)-derivation of a BCIK-algebra X. Then

1. Both $\theta(x)$ and $d_{(\theta,\phi)}(x)$ belong to the same branch for all $x \in X$.
2. $(\forall\, x \in X)\ (d_{(\theta,\phi)}(x \leq \theta(x))$.
3. $(\forall\, x, y \in X)\ (d_{(\theta,\phi)}(x) * \theta(y) \leq \theta(x) * d_{(\theta,\phi)}(y))$.

Proof.

1. For any $x \in X$, we get

$0 = d_{(\theta,\phi)} = d_{(\theta,\phi)}(a_x * x)$

$= (\theta(a_x) * d_{(\theta,\phi)}(x)) \wedge ((d_{(\theta,\phi)}(a_x) * \phi(x))$

$= ((d_{(\theta,\phi)}(a_x) * \phi(x)) * ((d_{(\theta,\phi)}(a_x) * \phi(x)) * (\theta(a_x) * d_{(\theta,\phi)}(x))$

Since $\theta(a_x) * d_{(\theta,\phi)}(x) \in L_p(X)$. Hence $\theta(a_x) \leq d_{(\theta,\phi)}(x)$, and so $d_{(\theta,\phi)} \in V(\theta(a_x))$.

2. Since $d_{(\theta,\phi)}$ is regular, $d_{(\theta,\phi)} = 0$. It follows from Lemma 7.3. that

$d_{(\theta,\phi)}(x) = \theta(x) \wedge d_{(\theta,\phi)}(x) \leq \theta(x)$.

3. Since $d_{(\theta,\phi)}(x) \leq \theta(x)$ for all $x \in X$, we have

$d_{(\theta,\phi)}(x) * \theta(y) \leq \theta(x) * \theta(y) \leq \theta(x) * d_{(\theta,\phi)}(y)$

If we take $\theta = \phi = f$ in proposition 7.6, then we have the following corollary.

Corollary 7.7: [6]. If d_f is a regular (r, l)-f-derivation of a BCIK-algebra X, then both f(x) and $d_f(x)$ belong to the same branch for all $x \in X$.

Now we provide conditions for an inside (or outside) (θ,ϕ)-derivation to be regular.

Theorem 7.8: Let $d_{(\theta,\phi)}$ be an inside (θ,ϕ)-derivation of a BCIK-algebra X. If there exists $a \in X$ such that $d_{(\theta,\phi)}(x) * \theta(a) = 0$ for all $x \in X$, then $d_{(\theta,\phi)}$ is regular.

Proof. Assume that there exists $a \in X$ such that $d_{(\theta,\phi)}(x) * \theta(a) = 0$ for all $x \in X$. Then

$0 = d_{(\theta,\phi)}(x*a) = ((d_{(\theta,\phi)}(x) * \theta(a)) \wedge \phi(x) * d_{(\theta,\phi)}(a))) * a$

$= (0 \wedge (\phi(x) * d_{(\theta,\phi)}(a)) * a = 0 * a$,

And so $d_{(\theta,\phi)}(0) = d_{(\theta,\phi)}(0 * x) = (d_{(\theta,\phi)}(0) * \theta(a)) = 0$. Hence $d_{(\theta,\phi)}$ is regular.

Theorem 7.9: If X is a BCIK-algebra, then every inside (or outside) (θ,ϕ)-derivation of X is regular.

Proof. Let $d_{(\theta,\phi)}$ be an inside (θ,ϕ)-derivation of a BCIK—algebra. Then

$d_{(\theta,\phi)}(0) = d_{(\theta,\phi)}(0 * x)$

$= (d_{(\theta,\phi)}(0) * \theta(x)) \wedge (\phi(0) \wedge d_{(\theta,\phi)}(x))$

$= (d_{(\theta,\phi)}(0) * \theta(x)) \wedge 0 = 0$.

If $d_{(\theta,\phi)}$ is an outside (θ,ϕ)-derivation of a BCIK—algebra X, then

$d_{(\theta,\phi)}(0) = d_{(\theta,\phi)}(0 * x)$

$= (\theta(0) * d_{(\theta,\phi)}(x)) \wedge (d_{(\theta,\phi)}(0) * \theta(x))$

$= 0 \wedge (d_{(\theta,\phi)}(0) * \theta(x)) = 0$.

Hence $d_{(\theta,\phi)}$ is regular.

To prove our results, we define the following notions:

Definition 7.10: For an inside (or outside) (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCIK-algebra X, we say that an ideal A of X, we say that an ideal A of X is a θ-ideal (resp. ϕ-ideal) if θ(A) $\subseteq$ A (resp. ϕ(A) $\subseteq$ A).

Definition 7.11: For an inside (or outside) (θ,ϕ)-derivation$_{(\theta,\phi)}$ of a BCIK-algebra X, we say that an ideal A of X, we say that an ideal A of X is $d_{(\theta,\phi)}$-invariant if $d_{(\theta,\phi)} \subseteq$ A.

Example 7.12: Let $d_{(\theta,\phi)}$ be an outside (θ,ϕ)-derivation ofX which is described Example 7.2. we know that A := {0,a} is both a θ -ideal and ϕ -ideal of X. But A := {0,a} is an ideal of X which is not $d_{(\theta,\phi)}$-invariant.

Theorem 7.13: Let $d_{(\theta,\phi)}$ be a outside (θ,ϕ)-derivation of a BCIK-algebra X. Then every θ -ideal of X is $d_{(\theta,\phi)}$-invariant.

Proof. Let A be a θ -ideal of X. Since $d_{(\theta,\phi)}$ is regular, it follows from Lemma 7.3 that $d_{(\theta,\phi)}$ = θ(x) $\wedge$ $d_{(\theta,\phi)}$(x) $\leq$ θ(x) for all x $\in$ X. Let y $\in$ X be such that y $\in$ $d_{(\theta,\phi)}$(A). Then y = $d_{(\theta,\phi)}$(x) for some x $\in$ A. Thus y * θ(x) = $d_{(\theta,\phi)}$(x) * θ(x) = 0 $\in$ A.

Note that θ(x) $\in$ θ(A) $\subseteq$ A. Since A is an ideal of X, it follows that y $\in$ A so that $d_{(\theta,\phi)}$(A) $\subseteq$ A. Therefore A is $d_{(\theta,\phi)}$-invariant.

If we take $\theta=\phi=1_X$ in Theorem 7.13. 1_X is the identity map, then we have the following corollary.

Corollary 7.14: [4]. Let d be a regular (r, l)-derivation of a BCIK-algebra X. Then every ideal of X is d-invariant.

If we take $\theta=\phi=$ f in Theorem 3.13, then we have the following corollary.

Corollary 7.15: [6]. Let d_f be a regular (r, l)-f-derivation of a BCIK-algebra X. Then every f-ideal of X is $d_{(\theta,\phi)}$-invariant.

Theorem 7.16: Let $d_{(\theta,\phi)}$ be an outside (θ,ϕ)-derivation of a BCIK-algebra X. If every θ -ideal of X is $d_{(\theta,\phi)}$-invariant, then $d_{(\theta,\phi)}$ is regular.

Proof. Assume that every θ-ideal of X is $d_{(\theta,\phi)}$-invariant. Since the zero ideal {0} is clearly θ -ideal and $d_{(\theta,\phi)}$-invariant, we have $d_{(\theta,\phi)}$({0}) $\subseteq$ {0}, and so

$d_{(\theta,\phi)}$ = 0. Hence $d_{(\theta,\phi)}$ is regular.

Combining Theorem 7.13. and 7.16., we have a characterization of a regular outside (θ,ϕ)-derivation.

Theorem 7.17: For an outside (θ,ϕ)-derivation$d_{(\theta,\phi)}$of a BCIK-algebra X, the following are equivalent:

1. $d_{(\theta,\phi)}$ is regular.
2. Every θ -ideal of X is $d_{(\theta,\phi)}$-invariant.

If we take $\theta=\phi=1_X$ in Theorem 3.17. where 1_X is the identity map, then we have the following corollary.

Corollary 7.18: [4]. Let d be an (r, l)-derivation of a BCIK-algebra X. Then d is regular if and only if every ideal of X is d-invariant.

If we take $\theta=\phi=$ f in Theorem 3.17, then we have the following corollary.

Corollary 7.19: [6]. For an (r, l)-f-derivation d_f of a BCIK-algebra X, the following are equivalent:

1. d_f is regular.
2. Every f-ideal of X is d_f-invarient.

CONCULUTION

In this present paper, we have consider the notions of regular inside (or outside) (θ,ϕ)-derivation, θ-ideal, ϕ-ideal and invariant inside (or outside) (θ,ϕ)-derivation of a BCIK-algebra, and investigated related properties. The theory of derivations of algebraic structures is a direct descendant of the development of classical Galosis theory. In our opinion, these definitions and main results can be similarly extended to some other algebraic system such as subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras and so forth.

In our future study the notion of regular (θ,ϕ)-derivation on various algebraic structures which may have a lot applications (θ,ϕ)-derivation BCIK-algebra, may be the following topics should be considered:

1. To find the generalized (θ,ϕ)-derivation of BCIK-algebra,
2. To find more result in (θ,ϕ)-derivation of BCIK-algebra and its applications,
3. To find the (θ,ϕ)-derivation of B-algebras, Q-algebras, subtraction algebras, d-algebra and so forth.

Acknowledgment

The author would like to thank Editor-in-Chief and referees for the valuable suggestions and corrections for the improvement of this paper.

References

[1] Y. Imai, K. Iseki, On axiom systems of propositional calculi XIV, proc. Japan Academy,42(1966), 19-22.

[2] Y.B. Jun and X.L. Xin On derivations of BCI – algebras, inform. Sci., 159(2004), 167-176.

[3] S Rethina Kumar, "Solvable Pseudo Commutators BCIK-Algebras', International Journal of Research Publication and Reviews Vol(3), Issue(2)(2021) Page 269-275, March 2021.

[4] S Rethina Kumar, "t-Regular t-Derivations On p-Semisimple BCIK-Algebras" EPRA International Journal of Multidisciplinary Research. Volume 7, Issue 3, pp.198-209, March 2021.

[5] S Rethina Kumar "PI-Lattices a Classical of bounded Commutative BCIK-Algebras" International Journal of Scientific Research in Mathematical Statistical Science, Recently Summited on March 2021.

[6] S Rethina Kumar "f-Regular f-Derivations on f-Semi simple BCIK-Algebra". International Journal of Multidisciplinary Research Review, Volume 7, Issue 3, pp37-56, March 2021.

[7] S Rethina Kumar "Regular Left Derivations On p-semi simple BCIK-algebra". International Journal of Trend in scientific Research and Development. Volume 5, Issue 3, pp559-602, March 2021.

[8] Generalizations of derivations in BCI-algebras, Appl. Math. Inf. Sci. 9 (2015).