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Regular Left Derivations on P-Semisimple Bcik-Algebras

S Rethina Kumar

Assistant Professor, Department of Mathematics, Bishop Heber College, Trichy, Tamil Nadu, India

ABSTRACT

In the present paper, we introduced the notion of left derivation of a BCIK – algebra and investigate some related properties. Using the idea of regular left derivation of a BCIK-algebra and investigate p-semi simple BCIK-algebra related properties. Using regular left derivation, we give characterizations of a regular left derivation on p-semi simple BCIK-algebra.

***KEYWORDS**: BCIK-algebra, p-semi simple, regular, left derivations*

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1. INTRODUCTION

In 1966, Y. Imai and K. Iseki [1,2] defined BCK – algebra in this notion originated from two different sources: one of them is based on the set theory the other is form the classical and non – classical propositional calculi. In [3]. Y.B. Jun and X.L. Xin applied the notion of derivation in ring and near – ring theory to BCI – algebras, and they also introduced a new concept called a derivation in BCI–algebras and its properties. In 2021[4], S Rethina Kumar introduce combination BCK–algebra and BCI–algebra to define BCIK–algebra and its properties and also using Lattices theory to derived some basic definitions, and they also the idea introduced a new concept called a regular derivation in BCIK-algebras. We give the Characterizations regular derivation p-semi simple algebra and its properties.

After the work of Jun and Xin[2004][3], many research articles have appeared on the derivations of BCI-algebra in different aspects as follows: In 2021[5], S Rethina Kumar have given the notation of t-derivation of BCIK-algebra and studied p-semi simple BCIK-algebra by using the idea of t-regular t-derivation in BCIK-algebra have extended the result of BCIK-algebra in the same paper they defined and studied the notion of left derivation of BCIK-algebras and investigated some properties of left derivation in p-semi simple BCIK-algebra

They investigated some of its properties defined a d-invariant ideal and gave conditions for an ideal to be d-invaroant. In non-commutative rings, the notation of derivations is extended to α-derivations, left derivations and central derivations. The properties of α-derivations and central derivations were discussed in several papers with respect to the ring structures. For left derivations, M. Bresar and J. Vukman[7] used them to give some results in prime and semi-prime rings. For skew polynomial rings, all left derivations are obtained in a similar way to a polynomial rings (see A. Nakajima and M. Sapanei[8], In 2021 [6], S Rethina Kumar introduced the notion of f-derivations of BCIK-algebra. The objective of this paper is to define left derivation on BCIK-algebra and then investigate a regular left derivation. Finally, we study left derivations on p-semisimple BCIK-algebra.

2. Preliminaries

Definition 2.1[4,5,6] BCIK algebra

Let X be a non-empty set with a binary operation * and a constant 0. Then (X, *, 0) is called a BCIK Algebra, if it satisfies the following axioms for all x, y, z Є X:

(BCIK-1) x*y = 0, y*x = 0, z*x = 0 this imply that x = y = z.

(BCIK-2) ((x*y) * (y*z)) * (z*x) = 0.

(BCIK-3) (x*(x*y)) * y = 0.

(BCIK-4) x*x = 0, y*y = 0, z*z = 0.

(BCIK-5) 0*x =0, 0*y = 0, 0*z = 0.

For all x, y, z Є X. An inequality ≤ is a partially ordered set on X can be defined x ≤ y if and only if

(x*y) * (y*z) = 0.

Properties 2.2: [4,5] I any BCIK – Algebra X, the following properties hold for all x, y, z Є X:

1. 0 Є X.
2. x*0 = x.
3. x*0 = 0 implies x = 0.

4. $0*(x*y) = (0*x) * (0*y)$.
5. $X*y = 0$ implies $x = y$.
6. $X*(0*y) = y*(0*x)$.
7. $0*(0*x) = x$.
8. $x*y \in X$ and $x \in X$ imply $y \in X$.
9. $(x*y) * z = (x*z) * y$
10. $x*(x*(x*y)) = x*y$.
11. $(x*y) *(y*z) = x*y$.
12. $0 \leq x \leq y$ for all $x, y \in X$.
13. $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$.
14. $x*y \leq x$.
15. $x*y \leq z \Leftrightarrow x*z \leq y$ for all $x, y, z \in X$
16. $x*a = x*b$ implies $a = b$ where a and b are any natural numbers (i. e)., $a, b \in N$
17. $a*x = b*x$ implies $a = b$.
18. $a*(a*x) = x$.

Definition 2.3: [4, 5, 6] Let X be a BCIK – algebra. Then, for all $x, y, z \in X$:
1. X is called a positive implicative BCIK – algebra if $(x*y) * z = (x*z) * (y*z)$.
2. X is called an implicative BCIK – algebra if $x*(y*x) = x$.
3. X is called a commutative BCIK – algebra if $x*(x*y) = y*(y*x)$.
4. X is called bounded BCIK – algebra, if there exists the greatest element 1 of X, and for any $x \in X$, $1*x$ is denoted by GG_x,
5. X is called involutory BCIK – algebra, if for all $x \in X$, $GG_x = x$.

Definition 2.4: [5, 6] Let X be a bounded BCIK-algebra. Then for all $x, y \in X$:
1. $G1 = 0$ and $G0 = 1$,
2. $GG_x \leq x$ that $GG_x = G(G_x)$,
3. $G_x * G_y \leq y*x$,
4. $y \leq x$ implies $G_x \leq G_y$,
5. $G_{x*y} = G_{y*x}$
6. $GGG_x = G_x$.

Theorem 2.5:[5] Let X be a bounded BCIK-algebra. Then for any $x, y \in X$, the following hold:
1. X is involutory,
2. $x*y = G_y * G_x$,
3. $x*G_y = y * G_x$,
4. $x \leq G_y$ implies $y \leq G_x$.

Theorem 2.6: [5] Every implicative BCIK-algebra is a commutative and positive implicative BCIK-algebra.

Definition 2.7: [5,10] Let X be a BCIK-algebra. Then:
1. X is said to have bounded commutative, if for any $x, y \in X$, the set $A(x,y) = \{t \in X : t*x \leq y\}$ has the greatest element which is denoted by $x \circ y$,
2. $(X, *, \leq)$ is called a BCIK-lattices, if $(X,\leq)$ is a lattice, where $\leq$ is the partial BCIK-order on X, which has been introduced in Definition 2.1.

Definition 2.8: [5] Let X be a BCIK-algebra with bounded commutative. Then for all $x, y, z \in X$:
1. $y \leq x \circ (y*x)$,
2. $(x \circ z) * (y \circ z) \leq x*y$,
3. $(x*y) * z = x*(y \circ z)$,
4. If $x \leq y$, then $x \circ z \leq y \circ z$,
5. $z*x \leq y \Leftrightarrow z \leq x \circ y$.

Theorem 2.9: [5,6] Let X be a BCIK-algebra with condition bounded commutative. Then, for all $x, y, z \in X$, the following are equivalent:
1. X is a positive implicative,
2. $x \leq y$ implies $x \circ y = y$,
3. $x \circ x = x$,
4. $(x \circ y) * z = (x*z) \circ (y*z)$,
5. $x \circ y = x \circ (y*x)$.

Theorem 2.10: [4,5,6] Let X be a BCIK-algebra.
1. If X is a finite positive implicative BCIK-algebra with bounded and commutative the $(X, \leq)$ is a distributive lattice,
2. If X is a BCIK-algebra with bounded and commutative, then X is positive implicative if and only if $(X, \leq)$ is an upper semi lattice with $x \vee y = x \circ y$, for any $x, y \in X$,
3. If X is bounded commutative BCIK-algebra, then BCIK-lattice $(X, \leq)$ is a distributive lattice, where $x \wedge y = y*(y*x)$ and $x \vee y= G (G_x \wedge G_y)$.

Theorem 2.11:[5] Let X be an involutory BCIK-algebra, Then the following are equivalent:
(1) $(X, \leq)$ is a lower semi lattice,
(2) $(X, \leq)$ is an upper semi lattice,
(3) $(X, \leq)$ is a lattice.

Theorem 2.12: [6] Let X be a bounded BCIK-algebra. Then:
1. every commutative BCIK-algebra is an involutory BCIK-algebra.
2. Any implicative BCIK-algebra is a Boolean lattice (a complemented distributive lattice).

Theorem 2.13: [5,6] Let X be a BCK-algebra, Then, for all $x, y, z \in X$, the following are equivalent:
1. X is commutative,
2. $x*y = x*(y*(y*x))$,
3. $x*(x*y) = y*(y*(x*(x*y)))$,
4. $x\leq y$ implies $x = y*(y*x)$.

Definition 2.14: Let X be a p-semi simple BCIK-algebra. We define addition + as $x + y = x*(0*y)$ for all $x, y \in X$. Then $(X,+)$ be an abelian group with identity 0 and $x - y = x*y$. Conversely, let $(X, +)$ be an abelian group with identity 0 and let $x - y = x*y$. Then X is a p-semi simple BCIK-algebra and $x + y = x*(0*y)$, for all $x, y \in X$ (see [5,6]). We denote x ⍰ y $= y * (y * x)$, $0 * (0 * x) = a_x$ and

$L_p(X) = \{a \in X / x * a = 0$ implies $x = a$, for all $x \in X\}$.

For any $x \in X$. $V(a) = \{a \in X / x * a = 0\}$ is called the branch of X with respect to a. We have

$x * y \in V (a * b)$, whenever $x \in V(a)$ and $y \in V(b)$, for all $x, y \in X$ and all $a, b \in L_p(X)$, for $0 * (0 * a_x) = a_x$ which implies that $a_x * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subset L_p(X)$ and $x * (x * a) = a$ and

$a * x \in L_p(X)$, for all $a \in L_p(X)$ and all $x \in X$.

Definition 2.15.([3,5]) Let X be a BCIK-algebra. By a (l, r)-derivation of X, we mean a self d of X satisfying the identity

$d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$.

If X satisfies the identity

$d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$,

then we say that d is a (r, l)-derivation of X

Moreover, if d is both a (r, l)-derivation and (r, l)-derivation of X, we say that d is a derivation of X.

Definition 2.16: ([3,5]) A self-map d of a BCIK-algebra X is said to be regular if $d(0) = 0$.

Definition 2.17: ([3,5]) Let d be a self-map of a BCIK-algebra X. An ideal A of X is said to be d-invariant, if $d(A) = A$.

3. Left Derivations

In this section, we define the left derivations

Definition 3.1; Let X be a BCIK-algebra by a left derivation of X, we mean a self-map D of X satisfying

$D(x * y) = (x * D(y)) \wedge (y * D(x))$, for all $x, y \in X$.

Example 3.2: Let $X = \{0,1,2\}$ be a BCIK-algebra with Cayley table defined by

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a map $D: X \longrightarrow X$ by

$$D(x) = \begin{cases} 2 \text{ if } x = 0,1 \\ 0 \text{ if } x = 2. \end{cases}$$

Then it is easily checked that D is a left derivation of X.

Proposition 3.3: Let D be a left derivation of a BCIK-algebra X. Then for all $x, y \in X$, we have

1. $x * D(x) = y * D(y)$.
2. $D(x) = a_{D(x)\wedge x}$.
3. $D(x) = D(x) \wedge x$.
4. $D(x) \in L_p(X)$.

Proof.

1. Let $x, y \in X$. Then
$D(0) = D(x * x) = (x * D(x)) \wedge (x * D(x)) = x * D(x)$.
Similarly, $D(0) = y * D(y)$. So, $D(x) = y * D(y)$.

2. Let $x \in X$. Then
$D(x) = D(x * 0)$
$= (x * D(0)) \wedge (0 * D(x))$
$= (0 * D(x)) * ((0 * D(x)) * (x * D(0)))$
$\leq 0 * (0 * (x * D(x))))$
$= 0 * (0 * (x * (x * D(x))))$
$= 0 * (0 * (D(x) \wedge x))$
$= a_{D(x)\wedge x}$.

Thus $D(x) \leq a_{D(x)\wedge x}$. But
$a_{D(x)\wedge x} = 0(0 * (D(x) \wedge x)) \leq D(x) \wedge x \leq D(x)$.
Therefore, $D(x) = a_{D(x)\wedge x}$.

3. Let $x \in X$. Then using (2), we have
$D(x) = a_{D(x)\wedge x} \leq D(x) \wedge x$.
But we know that $D(x) \wedge x \leq D(x)$, and hence (3) holds.

4. Since $a_x \in L_p(X)$, for all $x \in X$, we get $D(x) \in L_p(X)$ by (2).

Remark 3.4: Proposition 3.3(4) implies that D(X) is a subset of $L_p(X)$.

Proposition 3.5: Let D be a left derivation of a BCIK-algebra X. Then for all $x, y \in X$, we have

1. $Y * (y * D(x)) = D(x)$.
2. $D(x) * y \in L_p(X)$.

Proposition 3.6: Let D be a left derivation of a BCIK-algebra of a BCIK-algebra X. Then

1. $D(0) \in L_p(X)$.
2. $D(x) = 0 + D(x)$, for all $x \in X$.
3. $D(x + y) = x + D(y)$, for all $x, y \in L_p(X)$.
4. $D(x) = x$, for all $x \in X$ if and only if $D(0) = 0$.
5. $D(x) \in G(X)$, for all $x \in G(X)$.

Proof.

1. Follows by Proposition 3.3(4).

2. Let $x \in X$. From Proposition 3.3(4), we get $D(x) = a_{D(x)}$, so we have
$D(x) = a_{D(x)} = 0 * (0 * D(x)) = 0 + D(x)$.

3. Let $x, y \in L_p(X)$. Then
$D(x + y) = D(x * (0 * y))$
$= (x * D(0 * y)) \wedge ((0 * y) * D(x))$
$= ((0 * y) * D(x)) * (((0 * y) * D(x) * (x * D(0 * y)))$
$= x * D(0 * y)$
$= x * ((0 * D(y)) \wedge (y * D(0)))$
$= x * D(0 * y)$
$= x * (0 * D(y))$
$= x + D(y)$.

4. Let $D(0) = 0$ and $x \in X$. Then
$D(x) = D(x) \wedge x = x * (x * D(x)) = x * D(0) = x * 0 = x$.
Conversely, let $D(x) = x$, for all $x \in X$. So it is clear that $D(0) = 0$.

5. Let $x \in G(x)$. Then $0 * = x$ and so
$D(x) = D(0 * x)$
$= (0 * D(x)) \wedge (x * D(0))$
$= (x * D(0)) * ((x * D(0)) * (0 * D(x))$
$= 0 * D(x)$.

This give $D(x) \in G(X)$.

Remark 3.7: Proposition 3.6(4) shows that a regular left derivation of a BCIK-algebra is the identity map. So we have the following:

Proposition 3.8: A regular left derivation of a BCIK-algebra is trivial.

Remark 3.9: Proposition 3.6(5) gives that $D(x) \in G(X) \subseteq L_p(X)$.

Definition 3.10: An ideal A of a BCIK-algebra X is said to be D-invariant if $D(A) \subset A$.

Now, Proposition 3.8 helps to prove the following theorem.

Theorem 3.11: Let D be a left derivation of a BCIK-algebra X. Then D is regular if and only if ideal of X is D-invariant.

Proof.

Let D be a regular left derivation of a BCIK-algebra X. Then Proposition 3.8. gives that $D(x) = x$, for all $x \in X$. Let $y \in D(A)$, where A is an ideal of X. Then $y = D(x)$ for some $x \in A$. Thus

$Y * x = D(x) * x = x * x = 0 \in A$.

Then $y \in A$ and $D(A) \subset A$. Therefore, A is D-invariant.

Conversely, let every ideal of X Be D-invariant. Then $D(\{0\}) \subset \{0\}$ and hence D(0) and D is regular.

Finally, we give a characterization of a left derivation of a p-semisimple BCIK-algebra.

Proposition 3.12: Let D be a left derivation of a p-semisimple BCIK-algebra. Then the following hold for all x, y Є X:
1. D(x * y) = x * D(y).
2. D(x) * x = D(y) * Y.
3. D(x) * x = y * D(y).

Proof.
1. Let x, y Є X. Then
D(x * y) = (x * D(y)) ∧ ∧ (y * D(x)) = x * D(y).

2. We know that
(x * y) * (x * D(y)) ≤ D(y) * y and
(y * x) * (y * D(x)) ≤ D(x) * x.

This means that
((x * y) * (x * D(y))) * (D(y) * y) = 0, and
((y * x) * (y * D(x))) * (D(x) * x) = 0.

So
((x * y) * (x * D(y))) * (D(y) * y) = ((y * x) * (y * D(x))) * (D(x) * x). (I)

Using Proposition 3.3(1), we get,
(x * y) * D(x * y) = (y * x) * D(y * x). (II)

By (I), (II) yields
(x * y) * (x * D(y)) = (y * x) * (y * D(x)).

Since X is a p-semisimple BCIK-algebra. (I) implies that
D(x) * x = D(y) * y.

3. We have, D(0) = x * D(x). From (2), we get D(0) * 0 = D(y) * y or D(0) = D(y) * y.
So D(x) * x = y * D(y).

Theorem 3.13: In a p-semisimple BCIK-algebra X a self-map D of X is left derivation if and only if and if it is derivation.

Proof.
Assume that D is a left derivation of a BCIK-algebra X. First, we show that D is a (r,l)-derivation of X. Then
D(x * y) = x * D(y)
= (D(x) * y) * ((D(x) * Y) * (x * D(y)))
= (x * D(y)) ∧ (D(x) * y).

Now, we show that D is a (r,l)-derivation of X. Then
D(x * Y) = x * D(y)
= (x * 0) 8 D(y)
= (x * (D(0) * D(0)) * D(y)
= (x * ((x * D(x)) * (D(y) * y))) * D(y)
= (x * ((x * D(y)) * (D(x) * y))) * D(y)
= (x * D(y) * ((x * D(y)) * (D(x) * Y))
= (D(x) * y) ∧ (x * D(y)).

Therefore, D is a derivation of X.
Conversely, let D be a derivation of X. So it is a (r,l)-derivation of X. Then
D(x * y) = (x * D(y)) ∧ (D(x) * y)
= (D(x) * y) * ((D(x) * y) * (x * D(y)))
= x * D(y) = (y * D(x)) * ((y * D(x)) * (x * D(y)))
= (x * D(y)) ∧ (y * D(x)).

Hence, D is a left derivation of X.

4. Conclusion

Derivation is very interesting and important area of research in theory of algebraic structures in mathematics. The theory of derivations of algebraic structures is a direct descendant of the development of classical Galosis Theory. In this paper, we have considered the notation of left derivations in and investigated the useful properties of the left derivations in BCIK – algebra and its properties, the condition for left derivation to be regular is given. Finally, we give a characterization of a p-semisimple BCIK-algebra which admits derivation. Finally, we investigated the notation of left derivation in p-semi simple BCIK-algebra and established some results on left derivations in a p-semi simple BCIK-algebra. In our opinion, these definitions and main results can be similarly extended to some other algebraic system such as subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras and so forth.

In our future the generalized study of derivations of BCIK-algebras, may be the following topic should be considered:
1. To find the generalized left derivation BCIK-algebra and its properties.
2. To find the p-semisimple BCIK-algebra and its properties.
3. To find the more result in regular left derivation of p-semisimple BCIK-algebras and its applications
4. To find the regular left derivation of BCIK-algebra of B-algebras, Q-algebras, subtraction algebras, d-algebras and so forth

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