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RP-165: Solving Some Special Classes of Standard Cubic Congruence of Composite Modulus

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ABSTRACT

In this paper, a special class of standard cubic congruence of composite modulus is studied and formulated the solutions. The discussion is presented here. This cubic congruence has three types of solutions. In first case, it has exactly 3 incongruent solutions; in second case, it has nine incongruent solutions and in third case it has twenty-seven incongruent solutions. First time a suitable formulation is available for a class of standard cubic congruence. So, formulation is the merit of the paper.

***KEYWORDS**: Cubic Congruence, Composite Modulus, Cubic Residue, Formulation, Incongruent solutions*

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INTRODUCTION

If m is a positive integer, then the congruence $x^3 \equiv a\ (mod\ m)$ is called a standard cubic congruence of composite modulus. Also, if a is cubic residue of the modulus m, then the congruence is said to be solvable. If the congruence is solvable, then a can be written as $r^3 \equiv a\ (mod\ m), r\ being\ a\ residue\ of\ m.$

In this case the solvable Congruence can be written as: $x^3 \equiv r^3\ (mod\ m)$.

In this paper the author considered the congruence: $x^3 \equiv a^3\ (mod\ a^n)$ for formulation of its solutions in three different cases.

PROBLEM-STATEMENT

Here the problem is-"To formulate the solutions of the standard cubic congruence of the type:

$x^3 \equiv a^3 (mod\ m)\ in\ three\ different\ cases$

$Case - I: If\ a \neq 3l, l\ being\ a\ positive\ integer;$

$Case - II: If\ a = 3l, l\ being\ odd\ positive\ integer;$

$Case - III: If\ a = 3l, l\ being\ even\ positive\ integer$".

LITERATURE REVIEW

In the literature of mathematics, nothing is found about solving standard cubic congruence of prime and composite modulus. Only a definition is seen in the book of Zuckerman [1] and Thomas Koshy had defined only a cubic residue, page-548 [2]. David M Burton [3] in his book:"Elementary Number Theory", in the page no. 166, used the Theory of Indices to solve standard cubic congruence of prime modulus but established no formula for solutions. No pre-formulation is found for the congruence considered here. Only the author's formulations on standard cubic congruence of composite modulus are found in the literature of mathematics [4], [5], [6]. Here is one more standard cubic congruence of composite modulus is considered for formulation of solutions.

ANALYSIS & RESULTS

Consider the congruence $x^3 \equiv a^3 (mod\ m)$.

Case-I: Let $m = 3^n$, n positive integer $\&a \neq 3l, l\ being\ any\ positive\ integer$.

Then the congruence reduces to: $x^3 \equiv a^3 (mod\ 3^n)$.

For solutions, consider $x \equiv 3^{n-1}k + a\ (mod\ 3^n)$.

Then, $x^3 \equiv (3^{n-1}k + a)^3\ (mod\ 3^n)$

$\equiv (3^{n-1}k)^3 + 3.(3^{n-1}k)^2.a + 3.3^{n-1}k.a^2 + a^3\ (mod\ 3^n)$

$\equiv 3^{3n-3}k^3 + 3.3^{2n-2}k^2 + 3.3^{n-1}k + a^3 (mod\ 3^n)$

$\equiv 3^n k(3^{2n-3}k^2 + 3^{n-1}k + 1) + a^3 (mod\ 3^n)$

$\equiv a^3\ (mod\ 3^n)$

Thus it is seen that $x \equiv 3^{n-1}k + a\ (mod\ 3^n)$ satisfies the cubic congruence and hence must give all the solutions.

But it is seen that for $k = 3$, the solution formula reduces to:

$x \equiv 3^{n-1}.3 + a\ (mod\ 3^n)$

$\equiv 3^n + a \ (mod\ 3^n)$

$\equiv 0 + a \ (mod\ 3^n)$

This is the same solution as for $k = 0$.

Also if $k = 4 = 3 + 1$, then the solution formula reduces to:

$x \equiv 3^{n-1}.(3+1) + a \ (mod\ 3^n)$

$\equiv 3^n + 3^{n-1} + a \ (mod\ 3^n)$

$\equiv 3^{n-1} + a \ (mod\ 3^n)$

This is the same solution as for $k = 1$.

Therefore, all the solutions are given by:

$x \equiv 3^{n-1}k + a \ (mod\ 3^n); k = 0, 1, 2.$

This gives 3 *incongruent solutions of the congruence*.

Therefore, the result of this discussion is that the standard cubic congruence of composite modulus: $x^3 \equiv a^3 \ (mod\ 3^n)$ has 3 solutions given by:

$x \equiv 3^{n-1}k + a \ (mod\ 3^n); k = 0, 1, 2.$

Case-II: Let $a = 3l, l\ being\ an\ odd\ positive\ integer$.

Then the congruence reduces to:$x^3 \equiv (3l)^3 (mod\ 3^n)$.

For solutions, consider $x \equiv 3^{n-2}k + 3l \ (mod\ 3^n)$.

Then, $x^3 \equiv (3^{n-2}k + 3l)^3 \ (mod\ 3^n)$

$\equiv (3^{n-2}k)^3 + 3.(3^{n-2}k)^2.3l + 3.3^{n-2}k.(3l)^2 + (3l)^3 \ (mod\ 3^n)$

$\equiv 3^{3n-6}k^3 + 3.3^{2n-4}k^2.3l + 3.3^{n-2}k.(3l)^2 + (3l)^3 (mod\ 3^n)$

$\equiv 3^n k(3^{2n-6}k^2 + 3^{n-2}kl + 3l^2) + (3l)^3 (mod\ 3^n)$

$\equiv 0 + (3l)^3 \ (mod\ 3^n)$

Thus it is seen that $x \equiv 3^{n-2}k + 3l \ (mod\ 3^n)$ satisfies the cubic congruence and hence must give all the solutions.

But it is seen that for$k = 3^2 = 9$, the solution formula reduces to:

$x \equiv 3^{n-2}.3^2 + 3l \ (mod\ 3^n)$

$\equiv 3^n + 3l \ (mod\ 3^n)$

$\equiv 0 + 3l \ (mod\ 3^n)$

This is the same solution as for $k = 0$.

Also if $k = 10 = 3^2 + 1$, then the solution formula reduces to:

$x \equiv 3^{n-2}.(3^2 + 1) + 3l \ (mod\ 3^n)$

$\equiv 3^n + 3^{n-2} + 3l \ (mod\ 3^n)$

$\equiv 3^{n-2} + 3l \ (mod\ 3^n)$

This is the same solution as for $k = 1$.

Therefore, all the solutions are given by:

$x \equiv 3^{n-2}k + 3l \ (mod\ 3^n); k = 0, 1, 2, \dots \dots \dots \dots .8.$

This gives 9 *incongruent solutions of the congruence*.

Therefore, the result of this discussion is that the standard cubic congruence of composite modulus: $x^3 \equiv (3l)^3 \ (mod\ 3^n)$ has 9 incongruent solutions given by:

$x \equiv 3^{n-2}k + 3l \ (mod\ 3^n); k = 0, 1, 2 \dots \dots \dots .8.$

Case-III:Let $a = 3l, l\ being\ an\ even\ positive\ integer$.

Then the congruence reduces to:$x^3 \equiv (3l)^3 (mod\ 3^n)$.

For solutions, consider $x \equiv 3^{n-3}k + 3l \ (mod\ 3^n)$.

Then, $x^3 \equiv (3^{n-3}k + 3l)^3 \ (mod\ 3^n)$

$\equiv (3^{n-3}k)^3 + 3.(3^{n-3}k)^2.3l + 3.3^{n-3}k.(3l)^2 + (3l)^3 \ (mod\ 3^n)$

$\equiv 3^{3n-9}k^3 + 3.3^{2n-6}k^2.3l + 3.3^{n-3}k.(3l)^2 + (3l)^3 (mod\ 3^n)$

$\equiv 3^n k(3^{2n-9}k^2 + 3^{n-4}kl + 3l^2) + (3l)^3 (mod\ 3^n)$

$\equiv 0 + (3l)^3 \ (mod\ 3^n)$

Thus it is seen that $x \equiv 3^{n-3}k + 3l \ (mod\ 3^n)$ satisfies the cubic congruence and hence must give all the solutions.

But it is seen that for$k = 3^3 = 27$, the solution formula reduces to:

$x \equiv 3^{n-3}.3^3 + 3l \ (mod\ 3^n)$

$\equiv 3^n + 3l \ (mod\ 3^n)$

$\equiv 0 + 3l \ (mod\ 3^n)$

This is the same solution as for $k = 0$.

Also if $k = 28 = 3^3 + 1$, then the solution formula reduces to:

$x \equiv 3^{n-3}.(3^3 + 1) + 3l \ (mod\ 3^n)$

$\equiv 3^n + 3^{n-3} + 3l \ (mod\ 3^n)$

$\equiv 3^{n-3} + 3l \ (mod\ 3^n)$

This is the same solution as for $k = 1$.

Therefore, all the solutions are given by:

$x \equiv 3^{n-3}k + 3l \ (mod\ 3^n); k = 0, 1, 2, \dots \dots \dots \dots .26.$

This gives 27 *incongruent solutions of the congruence*.

Therefore, the result of this discussion is that the standard cubic congruence of composite modulus: $x^3 \equiv (3l)^3 \ (mod\ 3^n)$ has 27 incongruent solutions given by:

$x \equiv 3^{n-3}k + 3l \ (mod\ 3^n); k = 0, 1, 2 \dots \dots \dots .27.$

ILLUSTRATIONS:

Example-1: Consider the congruence $x^3 \equiv 8 \ (mod\ 81)$.

It can be written as $x^3 \equiv 2^3 (mod\ 3^4)$.

It is of the type $x^3 \equiv a^3 (mod\ 3^n)\ with\ a = 2, n = 4$.

It has exactly three incongruent solutions given by: $x \equiv 3^{n-1}k + a \ (mod\ 3^n), k = 0, 1, 2.$

$\equiv 3^{4-1}k + 2 \ (mod\ 3^4)$

$\equiv 27k + 2 \ (mod\ 81); k = 0, 1, 2.$

$\equiv 2, 29, 56 \ (mod\ 81)$.

These are the three solutions of the congruence.

Example-2: Consider the congruence $x^3 \equiv 729 \ (mod\ 2187)$.

It can be written as $x^3 \equiv 9^3 (mod\ 3^7)$.

It is of the type $x^3 \equiv a^3 (mod\ 3^n)\ with\ a = 9 = 3x3, n = 7$.

It has exactly nine incongruent solutions given by:

$x \equiv 3^{n-2}k + a \ (mod\ 3^n), k = 0, 1, 2, 3 \dots .8.$

$\equiv 3^{7-2}k + 9\ (mod\ 3^7)$

$\equiv 243k + 9\ (mod\ 81); k = 0, 1, 2, 3 \dots \dots 8.$

$\equiv 9, 252, 495, 738, 981, 1224, 1467, 1710, 1953\ (mod\ 2187)$

These are the nine incongruent solutions of the congruence.

Example-3: Consider the congruence $x^3 \equiv 216\ (mod\ 2187)$.

It can be written as $x^3 \equiv 6^3 (mod\ 3^7)$.

It is of the type $x^3 \equiv a^3 (mod\ 3^n)\ with\ a = 6 = 3x2, n = 7$.

It has exactly 27 incongruent solutions given by:

$x \equiv 3^{n-3}k + a\ (mod\ 3^n), k = 0, 1, 2, 3 \dots . .25, 26.$

$\equiv 3^{7-3}k + 6\ (mod\ 3^7)$

$\equiv 81k + 6\ (mod\ 2187); k = 0, 1, 2, 3 \dots \dots 25, 26.$

$\equiv 6, 87, 168, 249, 330, 411, \dots \dots \dots \dots .2031, 2112\ (mod\ 2187)$

These are the twenty-seven incongruent solutions of the congruence.

CONCLUSION

In the conclusion, it can be said that the standard cubic congruence: $x^3 \equiv a^3 (mod\ 3^n)$ has exactly three solutions given by $x \equiv 3^{n-1}k + a\ (mod\ 3^n); k = 0, 1, 2$ if $a \neq 3l$.

But if $a = 3l, l\ being\ an\ odd\ positive\ integer$, the cubic congruence has exactly nine solutions given by $x \equiv 3^{n-2}k + a\ (mod\ 3^n); k = 0, 1, 2, \dots \dots \dots \dots . . ,8$.

Also if $a = 3l, l\ being\ an\ even\ positive\ integer$, the cubic congruence has exactly twenty seven solutions given by: $x \equiv 3^{n-3}k + a\ (mod\ 3^n); k = 0, 1, 2, \dots \dots, 26$.

MERIT OF THE PAPER

The author's formulation of solutions of the cubic congruence under consideration made the finding of solutions easy and time-saving. A large number of solutions can be obtained in a short time with an easy efforts. Thus formulation of solutions is the merit of the paper.

REFERENCE

[1] Zuckerman H. S., Niven I., 2008, *An Introduction to the Theory of Numbers,* Wiley India, Fifth Indian edition, ISBN: 978-81-265-1811-1.

[2] Thomas Koshy, 2009, *Elementary Number Theory with Applications*, Academic Press, Second Edition, Indian print, New Dehli, India, ISBN:978-81-312-1859-4.

[3] David M Burton,2012, *Elementary Number Theory*, McGraw Hill education (Higher Education), Seventh Indian Edition, New Dehli, India, ISBN: 978-1-25-902576-1.

[4] B M Roy, *Formulation of standard cubic congruence of composite modulus modulo a product of odd primes and nth power of three*, International Journal of Engineering Technology Research & Management (IJETRM), ISSN: 2456-9348, Vol-04, Issue-10, Oct-20.

[5] B M Roy, *A review and reformulation of solutions of standard cubic congruence of composite modulus modulo an odd prime power integer*, International journal for scientific Development and research (IJSDR), ISSN: 2455-2631, Vol-05, Issue-12, Dec-20.

[6] B M Roy, *Solving some special standard cubic congruence modulo an odd prime multiplied by eight*, International Journal of Scientific Research and Development (IJSRED), ISSN:2581-7175, Vol-04, Issue-01, Jan-21.