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Compact Monothetic C-semirings

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ABSTRACT

In this article we present a brief history and some applications of semirings, the structure of compact monothetic c-semirings. The classification of these semirings be based on known description of discrete cyclic semirings and compact monothetic semirings.

KEYWORDS: *semigroup, semiring, topological semiring, cyclic semiring, monothetic semigroup, compact monothetic semigroup, compact monothetic semiring, c-semiring*

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INTRODUCTION

A semiring is an algebraic structure in which we can add and multiply elements, where multiplication distributes over addition.

The most familiar examples for semirings in classical algebra are the semiring of non-negative integers or the semiring of nonnegative real numbers with ordinary operations of addition and multiplication. The first examples of semirings appeared in the work of Dedekind [1]. But, then, it was the American mathematician Vandiver who used the term “semi-ring” in his 1934 paper for introducing an algebraic structure with two operations of addition and multiplication such that multiplication distributes on addition [2].

One of the classic areas of mathematics in which semirings arise is the theory of formal languages. A formal language is any set of words (finite strings of symbols) taken from a fixed finite set, called the alphabet. The set of all languages from a fixed alphabet can be viewed as a semiring where addition is the union of the languages and multiplication is the concatenation of the languages. Such semirings have applications in logic and theoretical computer science.

As a subset of all languages the set of regular languages is closed under union and concatenation, so is a semiring. By the famous result of Kleene, the regular languages are precisely the languages accepted by automata [3]. The operation of an automaton can be simulated by repeatedly multiplying the Boolean matrices that encode its rules [4]. This means that the study of regular languages boils down to considering finite collections of matrices over the Boolean semiring {0, 1} where addition is maximum and multiplication is minimum.

Tropical algebra is a relatively new area of mathematics which brings together ideas from algebra, order theory and discrete mathematics, and which has numerous applications in scheduling and optimization, formal language theory, numerical analysis and dynamical systems. Here the primary objects of study are the tropical semirings. The tropical semirings based on either the non-negative integer, the integer or real numbers, but with unusual semiring operations. Specifically, addition is either maximum (or minimum) and multiplication is usual addition.

BASIC DEFINITIONS AND PRELIMINARENT RESULTS

Definition 1: Semiring is an algebraic structure $< S,+,*>$ where S is a set, $< S,+>$ and $< S,*>$ are semigroups (i.e., both operations, addition “+"” and multiplication

" $*$ ", are associative) and those operations satisfy the laws of distributivity for both sides:

$a*(b+c)=a*b+a*c$ and

$(a+b)*c=a*c+b*c$, for any

$a,b,c\ \in S$.

The concept of semiring in this form was introduced by Vandiver in 1934 [2]. Sometimes we will suppose that a semiring S has the neutral element 0 (zero) relatively the addition or/and has the neutral element 1 (unit) relatively the multiplication and$0*x=x*0=0$ for any $x\in S$.

Definition 2: If $< S,* >$ is a group then a semiring $< S,+,*>$ is named semifield.

Definition 3: Semigroup S is named cyclic (monogenic) semigroup if it is generated by an element $a \in S$ (i.e, $S = \{a, a^2, \cdots, a^n, \cdots\}$) and this fact is denoted by $S = (a)$.

There exist two cases:
1. S is an infinite.
2. S is a finite $S = \{a, a^2, \cdots, a^k, \cdots, a^{k+n-1}\}$ where $a^k = a^{k+n}$. In this case S is named the cyclic semigroup of type (k, n).

Definition 4: Semiring $< S,+,*>$ is named (multiplicatively) cyclic semiring if$< S,*>$is a cyclic semigroup. If $< S,*>$ is generated by element $a \in S$ we will write $S = (a)$. If S has a unit 1 then $S = \{a^n \mid n \in N_0\}$ and $a^0 = 1$.

The semiring S with multiplicative semigroup of type (k, n) is named the cyclic semiring of type (k, n).

Cyclic semifield $\{a^k, \cdots, a^{k+n-1}\}$ is named cycle of the cyclic semiring of type (k, n) and denoted by C.

In this work we will consider only multiplicatively cyclic semirings. Thus, further we will omit the term multiplicatively.

Definition 5: A semiring S where $< S,+>$ is an idempotent semigroup (i.e., $x + x = x$ for any $x \in S$) is called idempotent semiring.

The operation of addition, called left (or right) if $x + y = x$ (or $x + y = y$) for any $x, y \in S$.

If $< S,+ >$ is a commutative idempotent semigroup then $< S,+ >$ becomes upper semilattices under the relation of order $x \le y \Leftrightarrow x + y = y$ for any $x, y \in S$.

Let S be a cyclic semiring $S = (a)$. There, is easy to identify that S is an idempotent semiring if $a + a = a$ (or $1 + 1 = 1$, if S has unit).

Let A bean algebraic system then the set $SubA$ of all subsystems of A form a lattice in relation to the inclusion. Some times, for certain systems, for instance, for the semigroups and semirings it is necessary to include an empty subsystem.

In the case of a topologic algebraic system A, of course we consider $\overline{Sub}\ A$ a lattice of all cloused subsystems of A.

Definition 6: Topological semiring S is a C-semiring if $\overline{Sub}\ S$ is a chain.

Now we need some previous results about semirings and cyclic semirings.

Theorem 1 [5]: Any finite semifield is isomorphic to the product of two semifield: one with a left addition and another with a right addition. Any finite semifield is idempotent.

Theorem 2 [5]: If a cyclic group G of order n ($|G| = n$) with generating element c is isomorphic the product of two groups G_1, and G_2 of order m and h respectively, then G_1 and G_2 are cyclic groups and m and h are coprimes, $(m, h) = 1$, $G_1 = (c^h)$, $G_2 = (c^m)$.

If C is a cyclic semifield, $C = (c)$, $|C| = n$ then $C = (C + e) \times (e + C)$, $|C + e| = m$, $|e + C| = h$, where e is an unit element of C, m and h are coprimes (i.e., $(m, h) = 1$), $C + e$ has a left addition, $e + C$ has right addition.

Theorem 3 [6]: Let S be a finite cyclic semiring $S = (a) \neq C$ then S has left addition or right addition or $a^s + a^t = a^q, q \ge \max\{s, t\}, s, t \in N$.

Let S be a finite cyclic semiring of type (k, n), $S \neq C$ and has non commutative addition.

Let's introduce a binary relation ρ on S:
$x\rho y \Leftrightarrow x, y \in C$ or $x = y$.

Definition 7: Let S = (a) be a semiring and ρ is a relation of equivalence on S. A relation ρ is a congruence on S if ρ preserve operations of S: aρb and cρd => (a + c)ρ(b + d) and (ac)ρ(bd) for all a, b, c, d ϵ S.

Definition 8: An element x_0 of semiring S is named absorbing if $x + x_0 = x_0 + x = x_0$ and $xx_0 = x_0x = x_0$ for all x, y ϵ S.

Theorem 4[6]: The relation ρ is a congruence. Factor semiring S/ρ is a finite cyclic semiring with absorbing element $[a^k]$ (where $[a^k]$ is the class of congruence ρ containing element a^k). A relation ρ identify the elements of C another classes are singleton. Addition in semiring S is left or right or commutative. If addition in S/ρ left (right) then addition in S is left (right) respectively.

MAIN RESULTS

Now we reduce the classification of compact monothetic C-semirings. Any semiring of this form is finite or infinite. At first we will describe finite monothetic C-semirings.

Lemma 1: Let S be a finite C-semiring of type (k, n) then $n = p^l$, where p is a prime number and <C, $*$ > is a p-primary cyclic group where C is the cycle of S.

Proof. Let S be a finite cyclic C-semiring. By Theorem 2, the cycle C of S is aC-semifield and $C = (C + e) \times (e + C)$, where $|C + e| = m$, $|e + C| = h$, m and h are coprime numbers and e is a unit of C.

If $m > 1$, and, $h > 1$, then susemirings $C + e$ and $e + C$ are not comparable. So, $C + e = e$ or $e + C = e$, for both cases of addition in C, (left or right).

Thus, Sub C is isomorphic a Subgr C (where Subgr C is a lattice of subgroups of <C, $*$>). Then,<C, $*$ > is a p-primary group [7].

Definition9: Let S be a semiring. Subset I of S is named ideal of S if for all a, b ϵ S and s ϵ S a + b, sa, as ϵ I.

Theorem 5[8]: Let S = (a) be a cyclic semiring of type (k, n). Subsets of form

$A_s = \{a^s, \ldots, a^k, \ldots, a^{k+n-1}\}$ where s ≤ k they and only they are ideals of S.

Definition 10: Let S = (a) be a finite cyclic semiring of type (k, n) and

$A_s = \{a^s, \ldots, a^k, \ldots, a^{k+n-1}\}$ where s ≤ k an ideal of S, then

$x\rho_A y \Leftrightarrow x, y \in A_s$ or $x = y$ for all x, y ϵ S.

Lemma 2: Let S be a finite cyclic semiring of type (k, n). The binary relation ρ_A on S is a congruence and S/ρ_A is a finite cyclic semiring of type (s, 1) with absorbing element $[a^s]$ ($[a^s]$ is the class of congruence ρ_A containing element a^s).

Proof. It is evident that $S/\rho_A = \{\{a\}, \{a^2\}, \dots, \{a^{s-1}\}, \{a^s, \dots, a^k, \dots, a^{k+n-1}\}\}$. Let's prove that ρ_A is a congruence. For operation of product the relation ρ_A be the same as congruence of Rice for semigroups.

Let's prove that ρ_A is a congruence for addition. Let be a, b, c, d ϵ S and $a\rho_A b$ and $c\rho_A d$.

If a, b ϵ $\{a^{t1}\}$, c, d ϵ $\{a^{t2}\}$ where t1, t2 ϵ {1, … , s-1} than a + c ϵ $[a^{t1} + a^{t2}]$ and

b + d ϵ $[a^{t1} + a^{t2}]$. This means that $(a + c)\rho_A(b + d)$.

If a, b ϵ $\{a^t\}$, c, d ϵ $A_s = \{a^s, \dots, a^k, \dots, a^{k+n-1}\}$ where t ϵ {1, … , s-1} than by Theorem 3$(a + c)\rho_A(b + d)$.

If a, b, c, d ϵ $A_s = \{a^s, \dots, a^k, \dots, a^{k+n-1}\}$ than by Theorem 3, a + c, b + d ϵ $A_s = \{a^s, \dots, a^k, \dots, a^{k+n-1}\}$ and $(a + c)\rho_A(b + d)$.

Therefore we proved that ρ_A is a congruence.

Lemma 3: Let S be a finite cyclic semiring of type (4, 1). Then, S is not a C-semiring.

Proof: In [9] obtained Cally's tables for addition for all cyclic semirings of type (4, 1).

Here, we can see that for these semirings S, the lattice Sub S isn't a chain as $a^2 + a^2 \neq a^3$. Thus, subsemiring (a^2) and (a^3) aren't comparable.

Lemma 4: Let S be a finite cyclic semiring of type (4, n). Than S is not a C-semiring.

Proof: Let S be a finite cyclic semiring of type (4, n). By lemma 2, the binary relation ρ_A on S is a congruence and S/ρ_A is a finite cyclic semiring of type (4, 1). Clearly, if S is a

C-semiring, than S/ρ_A is a C-semiring too. But, by lemma 3, S/ρ_A is not a C-semiring, than S is not a C-semiring.

Lemma 5: Let S be a finite cyclic semiring of type (k, 1) where k ≥ 4. Then S is not a C-semiring.

Proof. At the beginning, we will prove that if SubS is not a chain for all cyclic semirings S of type (k, 1) where k ≥ 4, then for all cyclic semirings S of type (k + 1, 1) SubS is not a chain too.

Assume that for all cyclic semirings S of type (k, 1) where k ≥ 4, SubS is not a chain and exist a cyclic C-semiring S of type (k + 1, 1). Let $x\rho_A y \Leftrightarrow$ x, y ϵ A_k or x = y for all x, y ϵ S. By lemma 2, the binary relation ρ_A on S is a congruence and $S/\rho_A$is a finite cyclic semiring of type (k, 1). If S is a cyclic C-semiring than S/ρ_A is a cyclic C-semiring, which is a contradiction.

By induction, all finite cyclic semirings S of type (k, 1) where k ≥ 4 are not a C-semirings.

Lemma 6: Let S be a finite cyclic semiring of type (k, n) where k ≥ 4. Then S is not a C-semiring.

Proof. Let S be a finite cyclic semiring of type (k, n) where k ≥ 4 and $x\rho_A y \Leftrightarrow$ x, y ϵ A_k or x = y for all x, y ϵ S. By lemma 2, the binary relation ρ_A on S is a congruence and S/ρ_A is a finite cyclic semiring of type (k, 1). If S is a finite cyclic C-semiring than S/ρ_A is a finite cyclic C-semiring, which is a contradiction.

From lemma 6 follows

Lemma 7: Let S be a finite cyclic C-semiring of type (k, n) then k ≤ 3.

Theorem 6: Let S be a finite cyclic semiring of type (k, n). S is a C-semiring if and only if k ≤ 3, $n = p^l$ (p is a prime) and if k = 3 then p ≠ 2.

Proof. By lemma 6, $n = p^l$ where p is a prime number. By lemma 12, k ≤ 3. If k = 3 then $|A_k| = |C| = p^l$ is odd number then p ≠ 2.

The sufficiency is evident.

Now, let consider the case when S is an infinite compact semiring.

Definition 11: The p-adic digit is an integer number between 0 and p – 1 (inclusive). A p-adic integer is a sequence $\{a_i\}$ i ϵ N_0 of p-adic digits. We write this sequence … a_s … $a_2a_1a_0$ (that is, the a_i are written from left to right). If n ϵ N_0 and n = $a_{k-1}a_{k-2}$ … a_1a_0 is its p-adic representation than we identify n with the p-adic integer $\{a_i\}$ i ϵ N_0 with a_i = 0 for i ≥ k.

(For example, 1 is the p-adic integer all whose digits are 0 except the right-most one which is 1 = … 0 … 001).

Definition 12: If α = $\{a_i\}$ i ϵ N_0 and β = $\{b_i\}$ i ϵ N_0 are two p-adic integer, we will define their sum. We define by induction a sequence $\{c_i\}$ i ϵ N_0 of

p-adic digits and a sequence $\{\varepsilon_i\}$ i ϵ N_0 of elements of {0, 1} as follows:

ε0 = 0;

c_i is $a_i + b_i + \varepsilon_i$ or $a_i + b_i + \varepsilon_i - p$

according as which of these two is a

p-adic digit

(in other words, is between 0 and p – 1).

In former case, $\varepsilon_{i+1} = 0$ and in the latter $\varepsilon_{i+1} = 1$.

Under those circumstances, we let α + β = $\{c_i\}$ i ϵ N_0.

Note that the rules described above are exactly the rules used for adding natural Let p be a prime number. The set of all p-adic integers forms a compact monothetic group with operation of addition. We will denote this group by Z_p.

Evident that Z_p has the generate element $b_0 = \{a_i\}$ i ϵ N_0 where $a_0 = 1$ and $a_i = 0$ for i ≥ 1.

That is, $Z_p = \overline{(\dots 0001)}$.

The lattice $\overline{Sub}Z_p$ is a chain:

$Z_p = G_0 > G_1 > G_2 > \dots > \{e\}$

.where $G_0 = \overline{(\dots 0001)}$, $G_1 = \overline{(\dots 0010)}$, $G_2 = \overline{(\dots 0100)}$, etc. Element e is the identity of Z_p.

Theorem 7: Let S be an infinite compact monothetic semiring $S = (a)$. S is a C-semiring if and only if:
1. $< S,* >$ is a group topologically isomorphic Z_p. S has left addition or right addition or $< S,+ >$ is a chain at relation of order $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in S$.
2. $< S,* >$ contains a compact monothetic group H topologically isomorphic Z_p and $S = \{a, a^2\} \cup H$ and $< S,* >$ is a compact monothetic semigroup of type ii) of Theorem 3, and $s \leq 2$. If $s = 2$ then $p \neq 2$. S has left addition or right addition or $< S,+ >$ is a chain at relation of order $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in S$.

addition or right addition or $< S, + >$ is a chain at relation of order $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in S$. In this case $\overline{Sub}\, S$ is isomorphic the lattice of cloused subsemigroups of $< S, * >$. Thus, Theorem 7 follows from the description of compact monothetic C-semigroups [10].

Here we will prove that if $S = \{a, a^2\} \cup H$ where H = Z_p and S is a C-semiring than

p ≠ 2. Really, if $S = \{a, a^2\} \cup H$and H = Z_2 than closed subsemirings

H = $\{b, b^2, \ldots, b^n, \ldots\}^*$ and $H_1 = \{a^2, b^2, b^3, \ldots, b^n, \ldots\}^*$ are not comparable. The element b is not belongs to H_1 and the element a^2 is not belongs to H. (Here X* denote the closure of X).

CONCLUSION

In the future, we should begin to study compact semirings S where $\overline{Sub}\, S$ is a chain.

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