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Application of Mathematics in Mechanical Engineering

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ABSTRACT
Applied Mathematics have been successfully used in the development of science and technology in 20th –21st century. In Mechanical Engineers, an application of Mathematics gives mechanical engineers convenient access to the essential problem solving tools that they use. In this paper, we will discuss some examples of applications of mathematics in Mechanical Engineering. We conclude that the role of mathematics in engineering remains a vital problem, and find out that mathematics should be a fundamental concern in the design and practice of engineering.

KEYWORDS: *Matrices, Laplace transform, Partial differential equation*

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INTRODUCTION

In this paper, several examples of applications of mathematics in mechanical engineering are discussed. Mathematics occupies a unique role in the Mechanical Engineering and represents a strategic key in the development of the technology. In this paper we elaborate some topics such as Matrices, Laplace transform, Partial differential equation for Mechanical Engineering.

SOME OF THE MATHEMATICAL TOOLS THAT ARE USED IN MECHANICAL ENGINEERING

- Matrices,
- Laplace transform
- Partial differential equation

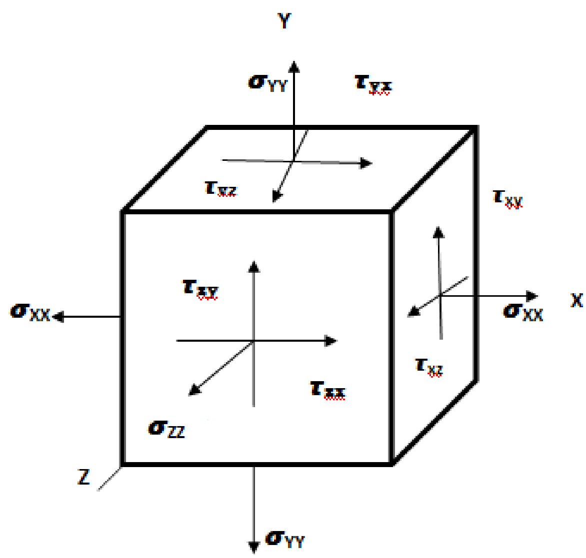
Matrices

Matrices: A rectangular arrangement of number, symbols, or expressions in rows and columns is known as matrix. Matrices play an important role in mechanical engineering syllabus. Some subjects mention below in which we will apply matrix knowledge –

In Engineering Material Sciences (Miller indices) matrix play an important role for defining crystal lattice geometries. In Strength of materials Strain matrix, stress matrix and the moment of inertia tensors are used for solving problems. We will also find application of matrices in analogous subjects like Design of Machine Elements, Design of Mechanical Systems. MATLAB stands for "Matrix Laboratory". Matrix is basic building block of MATLAB. Computer -aided Designing (CAD) cannot exist without matrices. In Robotics Engineering it is impossible to design a robot without the use of matrices. All the joint variables for forward/inverse kinematics and dynamics problems of the subject are noting down by matrix. Similarly, many concepts of matrices are used in Finite Element Analysis (FEA) and Finite Element Methods (FEM) for solving problems, just like CAD does. Mainly eigen value concept of matrices is used here.

These are some applications of matrices in mechanical engineering. Now we discuss a one example of matrix representing stress for calculation of principal stress.

INTRODUCTION TO THE STRESS TENSOR

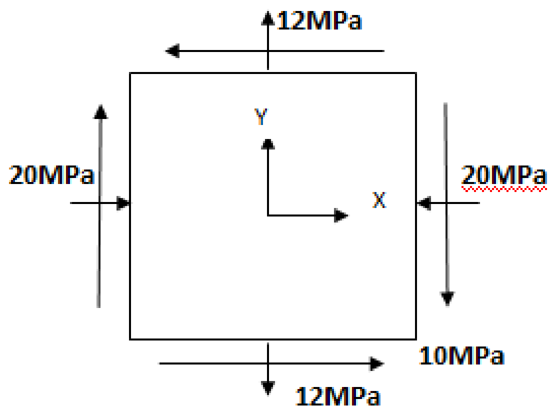


$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Here diagonal elements are normal stresses and off diagonal elements are shear stresses. On stress element in 3D, the normal and shear stresses can be compiled into a 3×3 matrix is called a stress tensor. From our observation, it is possible to find a set of three principal stresse for a given system. We know that, the shear stresses always become zero when principal stresses are acting. So in view of stress tensor, according to mathematical terms, this is the process of digonalization of matrix in which the eigen value of given matrix play the role of principal stresses.

Example

The state of plane stress at a point is represented by the stress element below. Find the principal stresses and angles at which the principal stresses act.



Let $A = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} = \begin{bmatrix} -20 & -10 \\ -10 & 12 \end{bmatrix}$

Now we calculate the eigen value of matrix A.

We consider the matrix form AX=$\boldsymbol{\lambda}$X, where I is identity matrix and λ become eigen value, X is eigenvector such that (A-$\boldsymbol{\lambda}$I)X=0.

To find the value of $\boldsymbol{\lambda}$ we consider Characteristic equation which is given by det(A-$\boldsymbol{\lambda}$I)=0

$$\begin{vmatrix} (-20-\lambda) & -10 \\ -10 & (12-\lambda) \end{vmatrix} = 0$$

$$\lambda^2 + 8 - 340 = 0$$

$$\lambda = 14.8679, -22.8679$$

So the principal stresses are 14.8679 and -22.8679 as discuss above. By using eigen values, we can calculate eigenvectors. These eigen vector are use for calculating the angle at which the principal stresses act.

Let X= $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector such that (A-$\boldsymbol{\lambda}$I)X=0

$$\begin{bmatrix} (-20-\lambda) & -10 \\ -10 & (12-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For

$\lambda = 14.8679$ $\begin{bmatrix} (-20-14.8679) & -10 \\ -10 & (12-14.8679) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} (-34.8679) & -10 \\ -10 & (-2.8676) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving above matrix, first eigenvector is given by $\begin{bmatrix} -0.2867 \\ 1 \end{bmatrix}$

For

$\lambda = -22.8679$ $\begin{bmatrix} (-20+22.8679) & -10 \\ -10 & (12+22.8679) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} (2.8679) & -10 \\ -10 & (34.8679) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving above matrix, second eigenvector is given by $\begin{bmatrix} 3.4867 \\ 1 \end{bmatrix}$

Now we calculate angles at which the principal stresses act, but before that we can check whether eigenvectors are correct or not. For this we will digonalized given matrix as fallow

D = B^{-1}AB

$= \begin{bmatrix} -0.265 & 0.924 \\ 0.265 & 0.0759 \end{bmatrix} \begin{bmatrix} -20 & -10 \\ -10 & 12 \end{bmatrix} \begin{bmatrix} -0.2867 & 3.4867 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 14.86 & 0 \\ 0 & -22.86 \end{bmatrix}$

Here B is model matrix (matrix combination of all eigenvector) and D is digonalized form of matrix A

These imply that the calculated eigenvector are correct.

Now to calculate angle, we must calculate unit eigenvector.

$\begin{bmatrix} -0.2867 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -0.2756 \\ 0.9612 \end{bmatrix}$ and $\begin{bmatrix} 3.4867 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0.9612 \\ 0.2756 \end{bmatrix}$

Now compile this unit eigenvector into rotation matrix R such that determinant R = +1

R= $\begin{bmatrix} 0.9612 & -0.2756 \\ 0.2756 & 0.9612 \end{bmatrix}$

Determinant R = (0.9612) (0.9612) + (0.2756) (0.2756) = 1

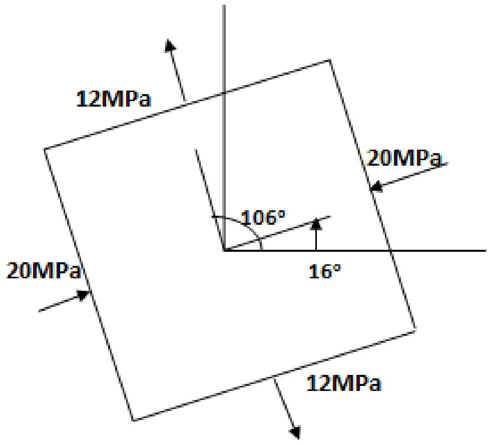
R= $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

D'=R^TAR

$= \begin{bmatrix} 0.9612 & 0.2756 \\ -0.2756 & 0.9612 \end{bmatrix} \begin{bmatrix} -20 & -10 \\ -10 & 12 \end{bmatrix} \begin{bmatrix} 0.9612 & -0.2756 \\ 0.2756 & 0.9612 \end{bmatrix}$

$= \begin{bmatrix} -22.86 & 0 \\ 0 & 14.86 \end{bmatrix}$

So θ = 16°, as we found earlier for one of the principal angles. Using the rotation angle of 16°, the matrix *A* (representing the original stress state of the element) can be transformed to matrix *D'* (representing the principal stress state).



In this way we can calculate the principal stresses and angles at which the principal stresses act by matrix method.

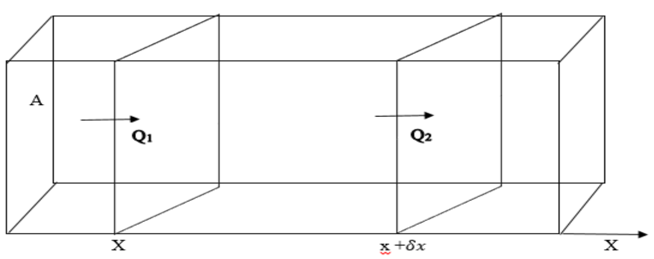
Partial differential equations

A large number of problems in fluid mechanics, solid mechanics, heat transfer, electromagnetic theory and other areas of physics and engineering science are modelled as Initial Value Problems and boundary value problems consisting of partial differential equations. In this paper, some of most important partial differential equations of one dimensional heat equation have been derived and solved.

- Partial differential equations are used for heat conduction analysis.
- Second order differential equation is used to find maxima and minima of function of several variables.
- Partial differential equation help to provide shape and interior, exterior design of machine.
- Partial differential equation is used to calculate heat flow in one and two dimensions.

One dimensional heat flow

Let us consider a conduction of heat along a bar whose both sides are insulated. Also the loss of heat from the sides of the bar by conduction or radiation is negligible. One end of the bar is taken as origin and direction of heat flow is along positive x-axis. The temperature u at any point of the bar is depend upon the distance x of the point from one end and the time t. The temperature of all points of any cross-section is the same.



Hence, the quantity of heat Q_1 flowing into the section at a distance x will be

$$Q_1 = -KA\left(\frac{\partial u}{\partial x}\right)_x \text{ Per second}$$

The negative sign on RHS is because u decreases as x increases,

The quantity of heat Q_2 flowing out of the section at a distance $x + \partial x$ will be

$$Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ Per second}$$

The amount of heat retained by the slab with thickness δx is, therefore,

$$Q_1 - Q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ Per second} \quad \ldots\ldots\ldots\ldots (1)$$

The rate of increase of heat in the slab is

$$= s\rho A\,\delta x\frac{\partial u}{\partial t}, \quad \ldots\ldots\ldots\ldots (2)$$

Where s is the specific heat and p is the density of the material of the bar.

From (1) and (2) we have therefore

$$s\rho A\delta x\frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\frac{\partial u}{\partial t} = \frac{K}{s\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2}, where\ C^2 = \frac{K}{s\rho}$$

is known as the thermal diffusivity of the material of the bar.

Solution of heat Equation

The heat equation is

$$\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2} \ldots\ldots\ldots\ldots 1$$

where the symbols have got their usual meanings.

Let $u = T(t)X(x), \ldots\ldots\ldots\ldots 2$

Then $\frac{\partial u}{\partial t} = X\frac{dT}{dt}$

And

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2}T$$

Taking the above substitutions in (1), we obtain

$$X\frac{dT}{dt} = C^2\frac{d^2X}{dx^2}T$$

$$\rightarrow \frac{1}{C^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = K(say) \ldots\ldots\ldots 3$$

The solutions will now be found under the following three cases:

Case I: When K=0, we have from (3)

$$\frac{1}{C^2 T}\frac{dT}{dt} = 0, \frac{1}{X}\frac{d^2X}{dx^2} = 0$$

$$\Rightarrow \frac{dT}{dt} = 0, \frac{d^2X}{dx^2} = 0$$

After integration we get
$\Longrightarrow T = C_1, and\ X = C_2 x + C_3$

Using this in equation (2), we get
$u = C_1(C_2 x + C_3) \ldots \ldots \ldots \ldots (A)$
Which is a solution of (1).

Case II: When K= m², i.e. K is >0, we have from equation (3)

$$\frac{1}{C^2 T}\frac{dT}{dt} = m^2, \frac{1}{X}\frac{d^2X}{dx^2} = m^2$$

$$\Rightarrow \frac{dT}{T} = m^2 C^2 dt, \frac{d^2X}{dx^2} - m^2 X = 0$$

$$\Longrightarrow \log T = m^2 C^2 t + \log C_4\ , A.E.\ D^2 - m^2 = 0$$

$$\Longrightarrow T = C_4 e^{m^2 C^2 t}, X = C.F. = C_5 e^{mx} + C_6 e^{-mx} (\because P.I. = 0)$$

From (2), we have therefore
$\Rightarrow u = C_4 e^{m^2 C^2 t (C_5 e^{mx} + C_6 e^{-mx})} \ldots \ldots \ldots \ldots \ldots (B)$

which is a solution of equation (1)
Case III: When K= - m², i.e. K < 0, we have from (3)

$$\frac{1}{C^2 T}\frac{dT}{dt} = -m^2, \frac{1}{X}\frac{d^2X}{dx^2} = -m^2$$

$$\Rightarrow \frac{dT}{T} = -m^2 C^2 dt, \frac{d^2X}{dx^2} + m^2 X = 0$$
$$\Longrightarrow \log T = -m^2 C^2 t + \log C_7\ ,$$

A.E. $D^2 + m^2 = 0 \Rightarrow D = \pm mi = \propto \pm \beta i \Longrightarrow \alpha = 0, \beta = m$
$\Longrightarrow T = C_7 e^{-m^2 C^2 t},$

X= C.F.$= e^{\alpha x}(C_8 \cos\beta x + C_9 \sin\beta x)$
$= e^{0x}(C_8 \cos mx + C_9 \sin mx)$
$= C_8 \cos mx + C_9 \sin mx\ (\because P.I. = 0)$

From (2) we have
$u = C_7 e^{-m^2 C^2 t}(C_8 \cos mx + C_9 \sin mx) \ldots \ldots \ldots \ldots \ldots (C)$
which is a solution of equation (1).

Among these solutions, we have to choose that solution which is consistent with physical nature of problem. Since u decreases as t increases, the only suitable solution of heat equation (1) is solution (C).

Example

Example: Find the temperature in a bar of length 2 units whose ends are kept at zero temperature and lateral surface insulated if the initial temperature is $\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}$.

Solution: One dimensional heat equation is

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \ldots \ldots \ldots \ldots (1)$$

The solution of equation (1) consistent with physical nature of problem is given by
$u = C_1 e^{-m^2 C^2 t}(C_2 \cos mx + C_3 \sin mx) \ldots \ldots \ldots (2)$

Where

$u(x,t) = 0\ at\ x = 0 \ldots \ldots \ldots \ldots \ldots (3)$
$u(x,t) = 0\ at\ x = 2 \ldots \ldots \ldots \ldots \ldots (4)$
$u(x,t) = \sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2}\ at\ t = 0 \ldots \ldots \ldots (5)$

Using the condition (3) in (2), we obtain
$0 = C_1 e^{-m^2 C^2 t}(C_2)$
$\Rightarrow C_2 = 0$

From (2), we have therefore
$u = C_1 e^{-m^2 C^2 t} C_3 \sin mx \ldots \ldots \ldots \ldots \ldots \ldots (6)$

Using condition (4) in (2), we obtain
$0 = C_1 e^{-m^2 C^2 t} C_3 \sin 2m$
$\Longrightarrow \sin 2m = 0 = \sin n\pi$
$\Longrightarrow 2m = n\pi$
$\Longrightarrow m = n\pi/2$

From (6) we have
$u = C_1 e^{-\left(\frac{n\pi}{2}\right)^2 C^2 t} C_3 \sin\left(\frac{n\pi x}{2}\right) \ldots \ldots \ldots \ldots \ldots \ldots (7)$

The general solution is

$$u = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{2}\right)^2 C^2 t} \sin\left(\frac{n\pi x}{2}\right) \ldots \ldots \ldots \ldots \ldots \ldots (8)$$

Using (5) in (8), we obtain

$$\sin\frac{\pi x}{2} + 3\sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$= b_1 \sin\left(\frac{\pi x}{2}\right) + b_2 \sin\left(\frac{2\pi x}{2}\right) + b_3 \sin\left(\frac{3\pi x}{2}\right) + b_4 \sin\left(\frac{4\pi x}{2}\right) + \ldots \ldots \ldots$$

$$\Rightarrow b_1 = 1 = b_5 = 3 = b_2 = b_3 = b_4 = b_6 = \cdots \ldots = 0$$

From (8), we have therefore

$$u = b_1 e^{-\frac{\pi^2}{4} C^2 t} \sin\frac{\pi x}{2} + b_5 e^{-\left(\frac{5\pi}{2}\right)^2 C^2 t} \sin\frac{5\pi x}{2}$$

$$u = e^{-\frac{\pi^2}{4} C^2 t} \sin\frac{\pi x}{2} + 3e^{-\left(\frac{5\pi}{2}\right)^2 C^2 t} \sin\frac{5\pi x}{2}$$

which is the required temperature.

Laplace Transform

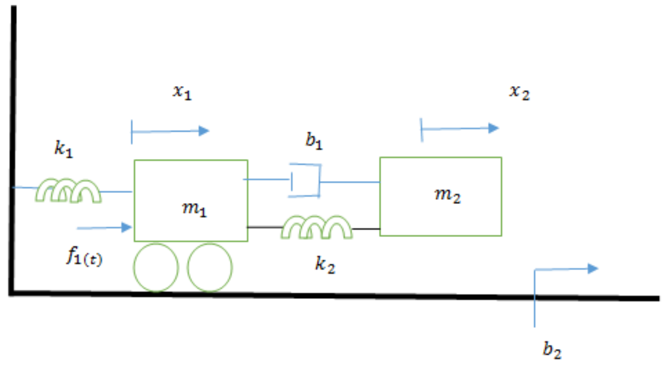
Laplace Transform: The Laplace Transform is the transform to time domain (t) into complex domain (s). Laplace Transform plays an important role in engineering system. The concept of Laplace Transform is applied in the area of science and technology such as to find the transfer function in mechanical system.

The Laplace Transform of f(t) is defined and denoted as

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$$

Where f(t) is the function of t and t > 0. Provided that integral exist, s is the parameter which may be real and complex.

Find the transfer function of the system shown below?



Transfer function: It is the relation between the output and the input of a dynamic system written in complex form (s). For a dynamic system with an input u(t) and an output y(t) ,the transfer function H(s) is the ratio between the complex representation (s) of the output Y(s) and input U(s).

Whenever they give any mechanical translation system, mass desk pot, spring will be in mechanical system. When we have to find out the transfer function of the system we need to take the output transfer by input transfer .Output is in terms of x_2 and input in terms f_1. If we consider $\frac{X_2(s)}{f_{1(s)}}$ we get transfer function of the system as fallow

First mass m_1, second mass m_2

Force due to the mass m_1 is $fm_1 = m_1 \frac{d^2x_1}{dt^2}$

Force due to the spring k_1 is $fk_1 = k_1x_1$

Force due to the desk pot b_1 is $fb_1 = b_1 \frac{d}{dt}(x_1 - x_2)$

Force due to the spring k_2 is $fk_2 = k_2(x_1 - x_2)$

External force is equal to the sum of all internal forces.

$$f_1(t) = fm_1 + fk_1 + fb_1 + fk_2$$

$$f_1(t) = m_1 \frac{d^2x_1}{dt^2} + k_1x_1 + b_1 \frac{d}{dt}(x_1 - x_2) + k_2(x_1 - x_2)$$

Taking Laplace transform both side, we get

$$F_1(s) = m_1 s^2 X_1(s) + k_1 X_1(s) + b_1 s (X_1(s) - X_2(s)) + k_2 (X_1(s) - X_2(s))$$

$$F_1(s) = (m_1 s^2 + b_1 s + k_1 + k_2) X_1(s) - (b_1 s + k_2) X_2(s).$$

Force acting on mass m_2 .

Force due to the mass m_2 *is* $fm_2 = m_2 \frac{d^2x_2}{dt^2}$

Force due to the desk pot b_1 is $fb_1 = b_1 \frac{d}{dt}(x_2 - x_1)$

Force due to the desk pot b_2 is $fb_2 = b_2 \frac{dx_2}{dt}$

Force due to the spring k_2 is $fk_2 = k_2(x_2 - x_1)$

No external force acting on mass m_2

$$0 = fm_2 + fb_1 + fb_2 + fk_2$$

$$0 = m_2 \frac{d^2x_2}{dt^2} + b_1 \frac{d}{dt}(x_2 - x_1) + b_2 \frac{dx_2}{dt} + k_2(x_2 - x_1)$$

Taking Laplace Transform both side

$$0 = m_2 s^2 X_2(s) + b_1 s \left(X_2(s) - X_1(s)\right) + b_2 s X_2(s) + k_2 (X_2(s) - X_1(s))$$

$$0 = X_2(s)(m_2 s^2 + b_1 s + b_2 s + k_2) - X_1(s)(b_1 s + k_2)$$

$$X_1(s) = \frac{(m_2 s^2 + b_1 s + b_2 s + k_2) X_2(s)}{(b_1 s + k_2)}$$

$$F_1(s) = (m_1 s^2 + b_1 s + k_1 + k_2) \frac{(m_2 s^2 + b_1 s + b_2 s + k_2) X_2(s)}{(b_1 s + k_2)} - (b_1 s + k_2) X_2(s)$$

$$F_1(s) = X_2(s) \left\{ \frac{(m_1 s^2 + b_1 s + k_1 + k_2)(m_2 s^2 + b_1 s + b_2 s + k_2) X_2(s) - (b_1 s + k_2)^2}{(b_1 s + k_2)} \right\}$$

$$\frac{X_2(s)}{F_1(s)} = \left\{ \frac{(b_1 s + k_2)}{(m_1 s^2 + b_1 s + k_1 + k_2)(m_2 s^2 + b_1 s + b_2 s + k_2) X_2(s) - (b_1 s + k_2)^2} \right\}$$

It is a transfer function of given mechanical system.

CONCLUSION

In this paper we conclude that mathematics is backbone in study of technical subject of Mechanical Engineering. Mathematics is applied in various field of mechanical engineering like maths is use in fluid mechanics, straight of material, machine design etc. In this way mathematical concept and procedure are used to solve problem in above mentioned field.

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