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# A Note on the Generalized Gamma Function

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**ABSTRACT**

In this paper, the generalized gamma functions of the first and second types are firstly introduced and investigated. It can be proven that the traditional gamma function is a special case of the first type of generalized gamma function. Besides, the iterative formula of the generalized gamma function will be fully derived. Finally, a numerical example is provided to illustrate the validity and effectiveness of our main result.

***KEYWORDS:*** *Generalized gamma function, gamma function, iterative formula, factorial function*

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## 1. INTRODUCTION

As we know, the gamma function first arose in regard to the interpolation problem for factorials. In conjunction with the factorial function, the gamma function is a generalization of the factorial function. In recent years, various gamma functions have been widely studied and explored; see, for example, [1-4] and the references therein. The above literatures show that gamma functions play a pivotal role in academic analysis and engineering applications. In this paper, generalized continuous functions of the first and second types will be firstly proposed. The purpose of this paper is to analyze the generalized gamma functions, and then derive the iterative formula of such functions. Finally, an example is provided to illustrate the applicability and validity of the main result.

## 2. PROBLEM FORMULATION AND MAIN RESULTS

Before presenting our main result, let us introduce generalized gamma function.

**Definition 1.** The generalized gamma function of the first type $G_c(a,\alpha)$ is defined by

$G_c(a,\alpha):=\int_0^\infty x^{\alpha-1}e^{-x}\cos ax\,dx$, with $\alpha>0$.

The generalized gamma function of the second type $G_s(a,\alpha)$ is defined by

$G_s(a,\alpha):=\int_0^\infty x^{\alpha-1}e^{-x}\sin ax\,dx$, with $\alpha>0$.

**Remark 1.** Note that the gamma function [1], defined by $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx$, can be regarded as a special case of the generalized gamma function of the first type in view of $\Gamma(\alpha)=G_c(0,\alpha)$.

**Lemma 1.** For any $a\in\Re$, one has

(1) $G_c(a,1)=\dfrac{1}{a^2+1}$;

(2) $G_s(a,1)=\dfrac{a}{a^2+1}$.

**Proof.** Two cases are separately discussed as follows.

Case 1: $(a=0)$

In this case, one can obtain

$G_c(0,1)=\int_0^\infty e^{-x}dx=1$ and

$G_s(0,1)=\int_0^\infty 0\,dx=0.$ (1)

Case 2: $(a\neq 0)$

Using the integration by parts, it can be obtained that

$$\int e^{-x}\cos ax\,dx=\frac{1}{a}e^{-x}\sin ax+\frac{1}{a}\int e^{-x}\sin ax;$$

$$\int e^{-x}\sin ax\,dx=-\frac{1}{a}e^{-x}\cos ax-\frac{1}{a}\int e^{-x}\cos ax.$$

It is easy to see that

$$G_c(a,1)=\frac{1}{a}G_s(a,1);$$

$$G_s(a,1)=\frac{1}{a}-\frac{1}{a}G_c(a,1).$$

It results

$$G_c(a,1)=\frac{1}{a^2+1} \text{ and } G_s(a,1)=\frac{a}{a^2+1}. \quad (2)$$

In summary, from (1) and (2), we conclude that $G_c(a,1)=\frac{1}{a^2+1}$ and $G_s(a,1)=\frac{a}{a^2+1}$. This completes our proof.

Now we present the recursive formula for the generalized gamma function.

**Theorem 1.** For any $a\in\Re$ and $\alpha>0$, one has

$$\begin{bmatrix} G_c(a,\alpha+1) \\ \hdashline G_s(a,\alpha+1) \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{a^2+1} & \frac{-a\alpha}{a^2+1} \\ \frac{a\alpha}{a^2+1} & \frac{\alpha}{a^2+1} \end{bmatrix} \cdot \begin{bmatrix} G_c(a,\alpha) \\ \hdashline G_s(a,\alpha) \end{bmatrix}.$$

**Proof.** Two cases are separately discussed as follows.
Case 1: $(a=0)$

In this case, using the integration by parts, one can obtain

$$G_c(0,\alpha+1)=\alpha\cdot G_c(0,\alpha), \quad (3a)$$

$$G_s(0,\alpha+1)=0. \quad (3b)$$

Case 2: $(a\neq 0)$
Using the integration by parts, it can be obtained that

$$\int x^\alpha e^{-x}\cos ax\,dx=\frac{1}{a}x^\alpha e^{-x}\sin ax$$
$$-\frac{\alpha}{a}\int x^{\alpha-1}e^{-x}\sin ax\,dx+\frac{1}{a}\int x^\alpha e^{-x}\sin ax\,dx;$$
$$\int x^\alpha e^{-x}\sin ax\,dx=-\frac{1}{a}x^\alpha e^{-x}\cos ax$$
$$+\frac{\alpha}{a}\int x^{\alpha-1}e^{-x}\cos ax\,dx-\frac{1}{a}\int x^\alpha e^{-x}\cos ax\,dx,$$

this implies

$$\begin{aligned}\int x^\alpha e^{-x}\cos ax\,dx &= \frac{\alpha}{a^2+1}\int x^{\alpha-1}e^{-x}\cos ax\,dx \\ &-\frac{a\alpha}{a^2+1}\int x^{\alpha-1}e^{-x}\sin ax\,dx \\ &-\frac{1}{a^2+1}x^\alpha e^{-x}\cos ax \\ &+\frac{a}{a^2+1}x^\alpha e^{-x}\sin ax\end{aligned}$$

and

$$\begin{aligned}\int x^\alpha e^{-x}\sin ax\,dx &= \frac{a\alpha}{a^2+1}\int x^{\alpha-1}e^{-x}\cos ax\,dx \\ &+\frac{\alpha}{a^2+1}\int x^{\alpha-1}e^{-x}\sin ax\,dx \\ &-\frac{a}{a^2+1}x^\alpha e^{-x}\cos ax \\ &-\frac{1}{a^2+1}x^\alpha e^{-x}\sin ax.\end{aligned}$$

It follows

$$\begin{aligned}G_c(a,\alpha+1) &= \frac{\alpha}{a^2+1}G_c(a,\alpha) \\ &-\frac{a\alpha}{a^2+1}G_s(a,\alpha)\end{aligned} \quad (4a)$$

and

$$\begin{aligned}G_s(a,\alpha+1) &= \frac{a\alpha}{a^2+1}G_c(a,\alpha) \\ &+\frac{\alpha}{a^2+1}G_s(a,\alpha).\end{aligned} \quad (4b)$$

This completes our proof, in view of (3) and (4).

Based on Lemma 1 and Theorem 1, we may readily obtain the following result.

**Corollary 1.**

$$\begin{bmatrix} G_c(a,n+1) \\ \hdashline G_s(a,n+1) \end{bmatrix} = A_n\cdot A_{n-1}\cdot A_{n-2}\cdot\cdots\cdot A_1\cdot\begin{bmatrix} \frac{1}{a^2+1} \\ \hdashline \frac{a}{a^2+1} \end{bmatrix}, \forall n\in \mathrm{N},$$

where

$$A_i := \begin{bmatrix} \frac{i}{a^2+1} & \frac{-ai}{a^2+1} \\ \frac{ai}{a^2+1} & \frac{i}{a^2+1} \end{bmatrix}, \quad \forall i\in \mathrm{N}.$$

## 3. ILLUSTRATIVE EXAMPLE

Consider the following definite integrals:

$$\int_0^\infty x^2e^{-x}\cos 2x\,dx \text{ and } \int_0^\infty x^2e^{-x}\sin 2x\,dx.$$

Thus, by Corollary 1 with $a=2$, it can be deduced that

$$\begin{bmatrix} \int_0^\infty x^2e^{-x}\cos 2x\,dx \\ \hdashline \int_0^\infty x^2e^{-x}\sin 2x\,dx \end{bmatrix} = \begin{bmatrix} G_c(2,3) \\ \hdashline G_s(2,3) \end{bmatrix}$$
$$= A_2A_1\begin{bmatrix} 1/5 \\ \hdashline 2/5 \end{bmatrix}$$
$$= \begin{bmatrix} -22/125 \\ \hdashline -4/125 \end{bmatrix}.$$

## 4. CONCLUSION

In this paper, the generalized gamma functions of the first and second types have been introduced and investigated. It can be proven that the traditional gamma function is a special case of the first type of generalized gamma function. Besides, the iterative formulas of the generalized gamma function have been fully derived. Finally, a numerical example has been provided to illustrate the validity and effectiveness of our main result.

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