International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 5 Issue 1, November-December 2020 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

A Note on the Generalization of the Mean Value Theorem

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ABSTRACT

In this paper, a new generalization of the mean value theorem is firstly established. Based on the Rolle's theorem, a simple proof is provided to guarantee the correctness of such a generalization. Some corollaries are evidently obtained by the main result. It will be shown that the mean value theorem, the Cauchy's mean value theorem, and the mean value theorem for integrals are the special cases of such a generalized form. We can simultaneously obtain the upper and lower bounds of certain integral formulas and verify inequalities by using the main theorems. Finally, two examples are offered to illustrate the feasibility and effectiveness of the obtained results.

KEYWORDS: *Rolle's theorem, Mean value theorem, Cauchy's mean value theorem, Mean value theorem for integrals, Generalized mean value theorem*

How to cite this paper: Yeong-Jeu Sun "A Note on the Generalization of the Mean Value Theorem" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-5 | Issue-1, December 2020, pp.1499-1501, URL: www.ijtsrd.com/papers/ijtsrd38258.pdf

IJTSRD38258

1. INTRODUCTION

In the past three decades, several kinds of mean valued theorems have been intensively investigated and proposed, such as mean value theorem (or the theorem of mean) [1], Cauchy's mean value theorem (or Cauchy's generalized theorem of mean) [2], mean value theorem for integrals [3], and others [3-10]. These theorems have many theoretical and practical applications including, but not limited to, maxima and minima, limits, inequalities, and definite integrals. Such theorems lead to very efficient methods for solving various problems and the importance of such theorems lie elsewhere [1-10]. It is worth mentioning that the proof of L'Hôpital's rule is based on Cauchy's mean value theorem.

In this paper, a simple generalized form of the mean value theorem will be investigated and established. It can be proven that the mean value theorem, the Cauchy's mean value theorem, and the mean value theorem for integrals are the special cases of such a generalized form. Based on such a generalized form, several kinds of generalized mean value theorems can be straightforwardly obtained. It will be shown that the main results can be applied to obtain the bounds of integrals. Meanwhile, we can prove inequalities by the main theorem.

2. PROBLEM FORMULATION AND MAIN RESULTS

Now we present the main result for the generalized form of the mean value theorem as follows.

Theorem 1. If $f_1, f_2, \cdots, f_n$ are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is a point η in (a,b) such that

$$\alpha_1 f_1'(\eta)+\alpha_2 f_2'(\eta)+\cdots+\alpha_n f_n'(\eta)=0,$$

where $(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\Re^n$ is any vector with

$$\sum_{i=1}^{n}\alpha_i\cdot[f_i(b)-f_i(a)]=0. \qquad (1)$$

Proof. Define

$$T(x):=\sum_{i=1}^{n}\alpha_i\cdot f_i(x). \quad (2)$$

Obviously, the function $T(x)$ is continuous in $[a,b]$ and differentiable in (a,b). In addition, one has

$$T(b)=\sum_{i=1}^{n}\alpha_i\cdot f_i(b)=\sum_{i=1}^{n}\alpha_i\cdot f_i(a)=T(a),$$

in view of (1) and (2). Therefore, by the Rolle's Theorem, there exists a number η in (a,b) such that $T'(\eta)=0$. It follows that

$$\sum_{i=1}^{n}\alpha_i\cdot f_i'(\eta)=0.$$

This completes our proof.

Define

$$W=\left\{t(\delta_1,\delta_2,\cdots,\delta_n)\in\Re^n \middle| t\in\Re\right\}$$

with $\delta_i:=f_i(b)-f_i(a), \forall i\in\{1,2,\cdots,n\}$. Obviously, W is a subspace of the Euclidean inner product space $\Re^n$. Thus, by Theorem 1, we may obtain an alternative form as follows.

Corollary 1. Let $f_1, f_2, \cdots, f_n$ are continuous in $[a,b]$ and differentiable in (a,b). There exists a point η on the open interval (a,b) such that

$$\alpha_1 f_1'(\eta)+\alpha_2 f_2'(\eta)+\cdots+\alpha_n f_n'(\eta)=0,$$
$$\forall\,(\alpha_1,\alpha_2,\cdots,\alpha_n)\in W^{\perp},$$

where $W^{\perp}$ is the orthogonal complement of W in $(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\Re^n$.

Simple setting $\frac{df_i(x)}{dx}=g_i(x), \forall\, i\in\{1,2,\cdots,n\}$ in Theorem 1, we may obtain the following generalized mean value theorem for integrals.

Corollary 2. If $g_1, g_2, \cdots, g_n$ are continuous in the open interval (a,b), then there is a point η in (a,b) such that

$$\alpha_1 g_1(\eta)+\alpha_2 g_2(\eta)+\cdots+\alpha_n g_n(\eta)=0,$$

where $(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\Re^n$ is any vector with

$$\sum_{i=1}^{n}\alpha_i\cdot\left[\int_a^b g_i(x)dx\right]=0.(3)$$

Based on the contra positive proposition of Theorem 1, we may obtain the following result.

Corollary 3. Let $f_1, f_2, \cdots, f_n$ be continuous in $[a,b]$ and differentiable in (a,b). If there exists a vector $(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\Re^n$ such that

$$\sum_{i=1}^{n}\alpha_i f_i'(\eta)\neq 0,\quad \forall\,\eta\in(a,b), \qquad (4)$$

then $\sum_{i=1}^{n}\alpha_i\cdot[f_i(b)-f_i(a)]\neq 0$.

Remark 1. By Theorem 1 with

$$f_2(x)=x,\quad \alpha_1=a-b,\quad \alpha_2=f_1(b)-f_1(a),$$

$$\alpha_3=\alpha_4=\cdots=\alpha_n=0,$$

in which case the condition (1) is obviously satisfied, it is easy to see that there exists a point η in (a,b) such that

$$(a-b)f_1'(\eta)+f_1(b)-f_1(a)=0.$$

This is exactly the same as the mean value theorem. Moreover, by Theorem 1 with

$$\alpha_1=f_2(b)-f_2(a),\quad \alpha_2=f_1(a)-f_1(b),$$

$$\alpha_3=\alpha_4=\cdots=\alpha_n=0,$$

in which case the condition (1) is apparently met, it can be shown that there exists a point η in (a,b) such that

$$[f_2(b)-f_2(a)]\cdot f_1'(\eta)+[f_1(a)-f_1(b)]\cdot f_2'(\eta)=0.$$

This result is exactly the same as the Cauchy's mean value theorem. Besides, by Corollary 2 with

$$g_2(x)=1,\quad \alpha_1=a-b,\quad \alpha_2=\int_a^b g_1(x)dx,$$

$$\alpha_3=\alpha_4=\cdots=\alpha_n=0,$$

in which case the condition (3) is evidently satisfied, it is easy to see that there exists a point η in (a,b) such that

$$(a-b)g_1(\eta)+\int_a^b g_1(x)dx=0,$$

this is exactly the same as the mean value theorem for integrals. Consequently, we conclude that our result is a nontrivial generalization of those results to the case with multiple functions.

3. ILLUSTRATIVE EXAMPLES

In this section, we provide two examples to illustrate the main results.

Example 1. Let $f_1(x)=x^2$, $f_2(x)=x^3$, $f_3(x)=x^4$, with the open interval $(a,b)=(0,1)$.

By Theorem 1, we conclude that there is a point η in $(0,1)$ such that

$$\alpha_1 f_1'(\eta)+\alpha_2 f_2'(\eta)+\alpha_3 f_3'(\eta)=0,$$
$$\forall\,(\alpha_1,\alpha_2,\alpha_3)\in\{(\alpha_1,\alpha_2,\alpha_3)\mid \alpha_1+\alpha_2+\alpha_3=0\}.$$

In case of $\alpha_1=\alpha_2=1, \alpha_3=-2$, we can see that

$$\alpha_1 f_1'\left(\frac{3+\sqrt{73}}{16}\right)+\alpha_2 f_2'\left(\frac{3+\sqrt{73}}{16}\right)$$
$$+\alpha_3 f_3'\left(\frac{3+\sqrt{73}}{16}\right)=0.$$

Similarly, in case of $\alpha_1=3, \alpha_2=-2, \alpha_3=-1$, we have

$$\alpha_1 f_1'\left(\frac{-3+\sqrt{33}}{4}\right)+\alpha_2 f_2'\left(\frac{-3+\sqrt{33}}{4}\right)$$
$$+\alpha_3 f_3'\left(\frac{-3+\sqrt{33}}{4}\right)=0.$$

Since $\frac{3+\sqrt{73}}{16}\in(0,1)$ and $\frac{-3+\sqrt{33}}{4}\in(0,1)$, Theorem 1 is verified.

The upper and lower bounds of some integral formulas can also be deduced by the main theorem.

Example 2. Prove that

$$-3\sqrt{2} \le \frac{\int_a^b g_1(x)dx}{\int_a^b g_2(x)dx} \le 3\sqrt{2}$$

if $g_1(x)$ and $g_2(x)$ are continuous in the open interval (a, b), with $g_2(x) > 0$ and

$$\frac{g_1(x)}{g_2(x)} = 3\sin 2x + 3\cos 2x, \quad \forall\, x \in (a, b).$$

Proof. By Corollary 2, there exists a point η in (a, b) such that

$$\frac{\int_a^b g_1(x)dx}{\int_a^b g_2(x)dx} = \frac{g_1(\eta)}{g_2(\eta)}$$

$$= 3\sin 2\eta + 3\cos 2\eta$$

$$= 3\sqrt{2}\sin\left(2\eta + \frac{\pi}{4}\right).$$

It follows that

$$-3\sqrt{2} \le \frac{\int_a^b g_1(x)dx}{\int_a^b g_2(x)dx} \le 3\sqrt{2}.$$

This completes our proof.

4. CONCLUSION

In this paper, a new generalization of the mean value theorem has been firstly proposed. Based on the Rolle's theorem, a simple proof has been provided to guarantee the correctness of such a generalization. Some corollaries have been evidently obtained by the main result. It has been shown that the mean value theorem, the Cauchy's generalized theorem of the mean, and the mean value theorem for integrals are the special cases of such a generalized form. Besides, we can simultaneously obtain the upper and lower bounds of certain integral formulas and verify inequalities by using the main theorems. Finally, two examples have been given to show the feasibility and effectiveness of the obtained results.

ACKNOWLEDGEMENT

The author thanks the Ministry of Science and Technology of Republic of China for supporting this work under grant MOST 109-2221-E-214-014. Besides, the author is grateful to Chair Professor Jer-Guang Hsieh for the useful comments.

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