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Nearly Regular Topological Spaces of the First Kind and the Second Kind

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ABSTRACT

In this paper we introduce the notions of nearly regular topological spaces of the first kind and the second kind and studies their properties. A number of important theorems regarding these spaces have been established.

KEYWORD: *Regular topological spaces, p-regular spaces, open set, closed set, product space, equivalence relation, projection mapping*

Mathematics Subject Classification: 54D10, 54D15, 54A05, 54C08.

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1. INTRODUCTION

This is the fourth in a series of our papers. The first, the second and the third such papers has appeared in 2018([13],[14],[15]). Regular topological spaces form a very important and interesting class of spaces in topology .The class of p-regular spaces is an example of generalization of this class ([8],[10]). Earlier, regular and normal topological spaces have been generalized in various other ways. p-regular, p-normal, β-normal and γ-normal spaces ([7], [8], [9], [10], [11]) are several examples of some of these.

In this paper we shall introduce a number of new important generalizations of regular spaces. We shall provide examples of such spaces and establish some of their important properties. The generalizations to be introduced by us in this paper is nearly regular topological spaces of the first kind and the second kind.

We have used the terminology and definitions of text book of S. Majumdar and N. Akhter [1], Munkres [2], Dugundji [3], Simmons [4], Kelley [5] and Hocking-Young [6].

We now define nearly regular spaces of the first kind and proceed to study them.

2. Nearly Regular Spaces of the First Kind

Definition 2.1: A topological space X will be called **nearly regular of the first kind (n. r .f. k.)** if there exists a nontrivial closed set F_0 such that for each $x \in X$, $x \notin F_0$, there exist disjoint open sets G and H such that $x \in G$ and $F_0 \subseteq H$.

Example 2.1: Let $X = \mathbb{R}, \Im = \langle \{\mathbb{R}, \emptyset, (1,2), (1,2)^c\} \cup A = \{\{x\} | x \in \mathbb{R} - (1,2)\} \cup B = \{\mathbb{R} - \{y_1, y_2\} | y_1 \in (1,2), y_2 \in (1,2)^c\} \rangle$

Here (1,2) is a closed set F_0. The points of A are the only points disjoint from F_0. Each of these points can be separated from F_0 by disjoint open sets. Let $x \in A$ then $\{x\}$ and F_0 are desired open sets. Then **X is n. r. f. k.** but **not regular.**

Let $F = (1,2)^c - [\{x_1\} \cup \ldots \ldots \cup \{x_n\}], x_i \in (1,2)^c$
$= (1,2)^c \cap [\{x_1\} \cup \ldots \ldots \cup \{x_n\}]^c$

Then F is closed, $x_1 \notin F$. x_1 and F cannot be separated by disjoint open sets.

Theorem 2.1: Every regular space is nearly regular space of the first kind but the converse is not true in general.

Proof: Let X be a regular space. Let F_0 be a closed subset of X and let $x \in X$ such that $x \notin F_0$. Now, since X is regular, there exist disjoint open sets G and H such that $x \in G$ and $F_0 \subseteq H$. Therefore X is nearly regular space of the first kind.

To see that the converse is always not true, let $X=\{a,b,c,d,e\}$, $\Im=\{X,\Phi,\{a,b\},\{a,b,e\},\{e\},\{a,b,c,d\}\}$ Then $(X,\Im)$ is a topological space in which the closed sets of X are X, Φ, $\{c,d,e\},\{e\},\{a,b,c,d\},\{c,d\}$.

The closed set $\{a,b,c,d\}$ and e can be separated by $\{a,b,c,d\}$ and {e}, but the closed set $\{c,d,e\}$ and a cannot be separated by disjoint open sets. Thus $(X,\Im)$ is nearly regular space of the first kind but not regular.

Theorem 2.2: A topological space X is nearly regular space of the first kind if and only if there exists a closed set F_0 such that for each $x\in X$ with $x\notin F_0$, there exists an open set G such that $x\in G\subseteq\overline{G}\subseteq F_0^{\,c}$.

Proof: First, suppose that X is nearly regular space of the first kind. Then there exists a closed set F_0 in X such that for each $x\in X$ with $x\notin F_0$, there exist open sets G and H such that $x\in G$, $F_0\subseteq H$ and $G\cap H=\phi$. It follows that $G\subseteq H^c\subseteq F_0^{\,c}$. Hence $G\subseteq\overline{G}\subseteq H^c\subseteq F_0^{\,c}$. Thus, $x\in G\subseteq\overline{G}\subseteq F_0^{\,c}$.

Conversely, suppose that there exists a closed subset F_0 of X such that for each $x\in X$ with $x\notin F_0$, there exist an open set G in X such that $x\in G\subseteq\overline{G}\subseteq F_0^{\,c}$. Let $\overline{G}^{\,c}$ = H. Then H is open, $G\cap H=\phi$ and $x\in G$ and $F_0\subseteq H$. Hence X is nearly regular space of the first kind.

Theorem 2.3: Let $\{X_i\}_{i\in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i\in I}X_i$ be the product space. If X_i nearly regular of the first kind, for each i, then X is nearly regular of the first kind.

Proof: Since each X_i is nearly regular of the first kind, there exists, for each i, a closed set F_i of X_i such that for each $x_i\in X_i$ with $x_i\notin F_i$ there are open sets U_i, V_i in X_i such that $x_i\in U_i$, $F_i\subseteq V_i$, $U_i\cap V_i=\phi$ ....(1)

Let F= $\prod_{i\in I}F_i$. Then F is closed in X. Let $\overline{x}\in X$ such that $\overline{x}\notin F$. Let $\overline{x}=\{\overline{x_i}\}$. Then there exists i_0 such that $\overline{x}_{i_0}\notin F_{i_0}$. By (1), there are open sets G_{i_0}, H_{i_0} in X_{i_0} such that $x_{i_0}\in G_{i_0}$, $F_{i_0}\subseteq H_{i_0}$, $G_{i_0}\cap H_{i_0}=\phi$. For each $j\in I$, $j\neq i_0$, let G_j and H_j be open sets in X_j such that $x_j\in G_j$, $F_j\subseteq H_j$. Then $G=\prod_{i\in I}G_i$, $H=\prod_{i\in I}H_i$ are open sets in X such that $x\in G$, $F\subseteq H$ and $G\cap H=\phi$. Therefore, X is nearly regular space of the first kind.

Theorem 2.4: Let $\{X_i\}_{i\in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i\in I}X_i$ be the product space. If X is nearly regular of the first kind, then at least one of the X_i's is nearly regular of the first kind.

Proof: Let X be a nearly regular space of the first kind. Then there exists a closed set F in X such that for every $x\in X$, $x\notin F$, there are open sets U and V in X such that $x\in U$ and $F\subseteq V$ and $U\cap V=\phi$. Let, for each $i\in I,\pi_i(F)=F_i$ where $\pi_i:X\to X_i$ is the projection map. Then each F_i is closed in X_i. For each $i\in I$, let $x_i\in X_i$ be such that $x_i\notin F_i$. Let $x=\{x_i\}$. Then $x\notin F$. Since X is nearly regular space of the first kind, there are open sets G and H in X such that $x\in G$, $F\subseteq H$ and $G\cap H=\phi$. Let $\pi_i(G)=G_i$, $\pi_i(H)=H_i$. Then G_i and H_i are open in X_i, for each $i\in I$. Since $G\cap H=\phi$ there exists $i_0\in I$ such that $G_{i_0}\cap H_{i_0}=\phi$. Clearly, $x_{i_0}\in G_{i_0}$, $F_{i_0}\subseteq H_{i_0}$. Hence X_{i_0} is nearly regular of the first kind.

Theorem 2.5: Every subspace of a nearly regular space of the first kind is nearly regular space of the first kind.

Proof: Let X be a nearly regular first kind space and Y a subspace of X. Since X is nearly regular first kind space, there exists a closed set F in X which can be separated from each point of X which is not contained in F.

Then for each $y\in Y\subseteq X$, $y\notin F$, there exist open sets U_1, U_2 in X such that $y\in U_1$, $F\subseteq U_2$ with $U_1\cap U_2=\phi$. Let $F_0=Y\cap F$. Then F_0 is closed in Y and clearly $y\notin F_0$. Also let $V_1=Y\cap U_1$, $V_2=Y\cap U_2$. Then V_1 and V_2 are disjoint open sets in Y where $y\in V_1$, $F_0\subseteq V_2$. Hence Y is nearly regular space of the first kind.

Corollary 2.1: Let X be a topological space and A, B are two nearly regular subspaces of X of the first kind. Then $A\cap B$ is nearly regular space of the first kind.

Proof: $A\cap B$ being a subspace of both A and B, $A\cap B$ is nearly regular space of the first kind by the above Theorem 2.5.

Theorem 2.6: Let X be a nearly regular T_1-space of the first kind and R is an equivalence relation of X. If the projection mapping p:X$\to\frac{X}{R}$ is closed then R is a closed subset of X×X.

Proof: We shall prove that R^c is open. So, let $(x, y) \in R^c$. It is sufficient to show that there exist two open sets G and H of X such that $x \in G$ and $y \in H$ and $G \times H \subseteq R^c$. For that $p(G) \cap p(H) = \phi$. Since $(x, y) \in R^c$, $p(x) \neq p(y)$ i.e; $x \notin p^{-1}(p(y))$. Again, since $\{y\}$ is closed, and since p is a closed mapping, p(y) is closed and since p is a continuous mapping, $p^{-1}(p(y))$ is closed. So by the nearly regularity of X of the first kind, there exist disjoint open sets G and U in X such that $x \in G$ and $p^{-1}(p(y)) \subseteq U$. Since p is a closed mapping, there exists an open set V containing p(y) such that $p^{-1}(p(y)) \subseteq p^{-1}(V) \subseteq U$. Writing $p^{-1}(V) = H$, we have $G \times H \subseteq R^c$.

Corollary 2.2: Let X be a nearly regular T_1-space of the first kind. R is an equivalence relation of X and $p: X \to \frac{X}{R}$ is closed and open mapping. Then $\frac{X}{R}$ is Hausdorff.

Proof: Since $p: X \to \frac{X}{R}$ is closed, by the proof of the above Theorem 2.6, R is a closed subset of X×X. Let $p(x)$ and $p(y)$ be two distinct points of $\frac{X}{R}$. Therefore $(x, y) \notin R$. Since R is a closed subset of X×X, there exist open sets G, H in X such that $x \in G$ and $y \in H$ and $G \times H \subseteq R^c$. So $p(x) \in p(G)$, $p(y) \in p(H)$. Since p is open, p(G) and p(H) are open sets of $\frac{X}{R}$ and since $G \times H \subseteq R^c$, $p(G) \cap p(H) = \phi$. Thus $\frac{X}{R}$ is Hausdorff.

We now define nearly regular spaces of the second kind and proceed to study them.

3. Nearly Regular Spaces of the Second Kind

Definition 3.1: A topological space X will be called **nearly regular of the second kind (n. r. s. k.)** if there exists a point $x_0 \in X$ such that for each nontrivial closed set F in X with $x_0 \notin F$, there exist disjoint open sets G and H such that $x_0 \in G$ and $F \subseteq H$.

Example 3.1: Let x_0 be a point in $\mathbb{R}^n$ such that for every closed set F in $\mathbb{R}^n$, $x_0 \notin F$. Since $\mathbb{R}^n$ is T_1, $\{x_0\}$ is closed and since $\mathbb{R}^n$ is normal and F is closed, $\{x_0\}$ and F can be separated by disjoint open sets. Thus $\mathbb{R}^n$ is **nearly regular of the second kind**.

Example 3.2: The Example 2.1 of **n. r. f. k.** is **not n. r. s. k.** The closed sets of the form:
(1,2), $(1,2)^c$, $\mathbb{R} - \{x_{n_1}, \dots \dots, x_{n_k}\}$, $\{y_1, y_2\}$.

Let $z \in \mathbb{R}$. Let $F = \{y_1, y_2\}$. If $z \in (1,2)$, then z and F can be separated by disjoint open sets.

If $z \notin (1,2)$, then $z = y_2$ for some $\mathbb{R} - \{y_1, y_2\} \in B$. So, z cannot be separated from $F = \{y_1, y_2\}$.

Theorem 3.1: Every regular space is nearly regular space of the second kind but the converse is not true in general.

Proof: Let X be a regular space. Let x_0 be a point in X and let F be a closed subset in X such that $x_0 \notin F$. Now, since X is regular, there exist disjoint open sets G and H such that $x_0 \in G$ and $F \subseteq H$. Therefore X is nearly regular of the second kind.

To see that the converse is always not true,
let $X = \mathbb{R}, \Im = \langle\{\mathbb{R}, \ , \{x_0\}, \{x_0\}^c\} \cup \{(n, n+1)^c | n \in \mathbb{N}\}, x_0 \notin \mathbb{N}\rangle$

Let $x_0 \notin (n_0, n_0+1)$. The closed sets are finite unions of $\{x_0\}$ and $(n, n+1), n \in \mathbb{N}$. x_0 and (n_0, n_0+1) are separated by $\{x_0\}$ and $\{x_0\}^c$. Thus X is **n. r. s. k.**

But X is not regular. Because if x=5, and F=(5, 6), then F is closed and $x \notin F$. But x and F cannot be separated by disjoint open sets.

Theorem 3.2: A topological space X is nearly regular space of the second kind if and only if there exists a point x_0 in X such that for each nontrivial closed set F in X with $x_0 \notin F$, there exists an open set G such that $x_0 \in G \subseteq \overline{G} \subseteq F^c$.

Proof: First, suppose that X be a nearly regular space of the second kind. Then there exists a point $x_0 \in X$ such that for each nontrivial closed set F in X with $x_0 \notin F$, there exist open sets G and H such that $x_0 \in G$, $F \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F^c$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F^c$. Thus, $x_0 \in G \subseteq \overline{G} \subseteq F^c$.

Conversely, suppose that there exists a point x_0 in X such that for each nontrivial closed set F in X with $x_0 \notin F$, there exists an open set G such that $x_0 \in G \subseteq \overline{G} \subseteq F^c$. Let $\overline{G}^c$ = H. Then H is open, $G \cap H = \phi$ and $x_0 \in G$ and $F \subseteq H$. Hence X is nearly regular space of the second kind.

Theorem 3.3: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let $X = \prod_{i \in I} X_i$ be the product space. If X_i nearly regular of the second kind, for each i, then X is nearly regular of the second kind.

Proof: Since each X_i is nearly regular of the second kind, there exists, for each i, a point x_i in X_i such that for each closed subset F_i in X_i with $x_i \notin F_i$ there are open sets U_i, V_i in X_i such that $x_i \in U_i$, $F_i \subseteq V_i$, $U_i \cap V_i = \phi$ …………………………(1)

Let F= $\prod_{i\in I} F_i$. Then F is closed in X. Let $x \in X$ such that $x \notin F$. Let $x = \{x_i\}$. Then there exists i_0 such that $x_{i_0} \notin F_{i_0}$. By (1), there are open sets G_{i_0}, H_{i_0} in X_{i_0} such that $x_{i_0} \in G_{i_0}$, $F_{i_0} \subseteq H_{i_0}$, $G_{i_0} \cap H_{i_0} = \phi$. For each $j \in I$, $j \neq i_0$, let G_j and H_j be open sets in X_j such that $x_j \in G_j$, $F_j \subseteq H_j$. Then $G = \prod_{i\in I} G_i$, $H = \prod_{i\in I} H_i$ are open sets in X such that $x \in G$, $F \subseteq H$ and $G \cap H = \phi$. Therefore, X is nearly regular space of the second kind.

Theorem 3.4: Let $\{X_i\}_{i\in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i\in I} X_i$ be the product space. If X is nearly regular of the second kind, then at least one of the X_i's is nearly regular of the second kind.

Proof: The proof of the Theorem 3.4 of the above is almost similar to the proof of the Theorem 2.4.

Theorem 3.5: Any subspace of a nearly regular space of the second kind is nearly regular space of the second kind.

Proof: The proof of the Theorem 3.5 follows from the proof of the Theorem 2.5.

Corollary 3.1: Let X be a topological space and A, B are two nearly regular subspaces of X of the second kind. Then $A \cap B$ is nearly regular space of the second kind.

Proof: The proof of the Corollary 3.1 of the above is almost similar to the proof of the Corollary 2.1.

Theorem 3.6: Let X be a nearly regular T_1-space of the second kind and R is an equivalence relation of X. If the projection mapping p:X $\rightarrow \frac{X}{R}$ is closed. Then R is a closed subset of X×X.

Proof: The proof of the Theorem 3.6 is most similar to the proof of the Theorem 2.6.

Corollary 3.2: Let X be a nearly regular T_1-space of the second kind. R is an equivalence relation of X and p:X $\rightarrow \frac{X}{R}$ is closed and open mapping. Then $\frac{X}{R}$ is Hausdorff.

Proof: The proof of the Corollary 3.2 follows from the proof of the Corollary 2.2.

Theorem 3.7: Every metric space is both n. r. f. k. and n. r. s. k.

Proof: Since every metric space is regular, therefore, it is n. r. f. k. and n. r. s. k.

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