International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 5 Issue 1, November-December 2020 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

A Study of Closure of An Operator

Praveen Sharma

Student, Department of Mathematics, University of Delhi, New Delhi, India

ABSTRACT

In this paper first we generalizes the Hilbert-Adjoint of a linear operator and showed that it is always closed for any linear operator with the condition that the domain of the operator is dense.

We also proved that "Let *J* be a closed operator defined in *H* with dense domain then *D(J*)* is dense and *J** = J* . "

We also proves Closed graph theorem for complex Hilbert spaces as a corollary of our results.

Keywords: *Separable, Closable, Closure, Operator extension.*

How to cite this paper: Praveen Sharma "A Study of Closure of An Operator" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-5 | Issue-1, December 2020, pp.347-353, URL: www.ijtsrd.com/papers/ijtsrd37914.pdf

IJTSRD37914

INTRODUCTION

In this Paper we take *H* as the complex Hilbert space and *D(J)* denotes the domain of a linear operator *J*.

In section 1 we generalizes the Hilbert- Adjoint for any linear operator with dense domain in *H* and prove

"If *J* is defined everywhere on *H*. Then its Hilbert-adjoint is bounded."

In section 2 we first define closable operators and show with the help of an example that every operator need not closable and if it is closable then graph of its closure is equals to the closure of its graph.

In Section 3 we proved our result
"Let J be a closed operator defined in *H* with dense domain then *D(J*)* is dense and *J** = J*."

As a result of which Closed graph Theorem for Complex Hilbert Spaces comes out as Corollary.

1 Hilbert-Adjoint

1.1 Definition: Let $J : H \to H$ be a bounded operator then its Hilbert adjoint always exists [2] and is also a bounded linear operator J^* defined everywhere on H such that

$$(J\varphi,\psi) = (\varphi,J^*\psi)\ \forall\ \varphi,\psi \in H$$

or

$$(J\varphi,\psi) = (\varphi,\eta) \quad \text{and} \quad J^*\psi = \eta \ (1)$$

1.2 Remark: We can use (1) to generalizes the Hilbert Adjoint of any operator.

1.3 Lemma: Let $J : D(J) \to H$ be any operator then $J^*\psi = \eta$ in (1) is unique iff

$\overline{D(J)} = H$. (i.e the domain of *J* is dense in H.)

Proof: Let $J^*\psi = \eta$ in (1) is unique. Suppose $\overline{D(J)} \neq H$

⇒ ∃ 0 6= μ ∈ H such that

$$\mu \perp \overline{\mathcal{D}(\mathcal{J})} \Rightarrow \mu \perp \mathcal{D}(\mathcal{J})$$
$$\Rightarrow (\phi,\mu) = 0\ \forall\ \phi \in \mathcal{D}(J)$$

From (1) we have

$$(J\varphi,\psi) = (\varphi,\eta) + 0$$
$$= (\varphi,\eta) + (\varphi,\mu)$$
$$= (\varphi,\eta + \mu)\ \forall\ \varphi \in D(J)$$

Which contradicts the uniqueness of $J : D(J) \to H$. Hence $\overline{D(J)} = H$.

Conversely

Let the domain of *J* is dense in H. Then

$$(J\phi,\psi) = (\phi,\eta)$$
$$= (\phi,\eta+\mu)\ \forall\ \phi \in \mathcal{D}(J)$$
$$\Rightarrow (\phi,\mu) = 0\ \forall\ \phi \in \mathcal{D}(J)$$
$$\Rightarrow \mu \perp \mathcal{D}(J) \Rightarrow \mu \perp \overline{\mathcal{D}(J)} = \{0\}$$

Hence unique.

1.4 Remark: Lemma 1.3 shows that Hilbert-Adjoint of any linear operator *J* exists iff the domain of *J* is dense. Now using this Lemma we generalizes the definition of Hilbert-Adjoint for any linear operator.

1.5 Definition Let $J : D(J) \to H$ be any linear operator whose domain is dense the its Hilbert-Adjoint exists and is a linear operator $J^* : D(J^*) \to H$ such that $D(J^*)$ contains all $\psi \in H$ such that $\exists\ \eta$ with $(J\varphi,\psi) = (\varphi,\eta)\ \forall\ \varphi \in D(T)$ and $J^*\psi = \eta$

Proposition 1.6: Let J be an operator defined everywhere on H.Then J^* is bounded.

Proof: Suppose J^* is unbounded. Then $\exists$ a sequence (v_n) each of norm 1 and $||J^*v_n|| \to \infty$ as $n \to \infty$ For each n define a functional P_n on H by $P_n(\varphi) = (J\varphi,v_n) = (\varphi,J^*v_n)$ By Schwarz Lemma [2] we have

$$|P_n(\varphi)| = |(\varphi,J^*v_n)| \le ||\varphi||\,||J^*v_n||\ \forall\ \varphi \in H$$

$\therefore P_n$ is bounded for all n. For each $\varphi \in H$ again by Schwarz Lemma we have

$$|P_n(\varphi)| = |(J\varphi,v_n)| \le ||J\varphi||\ \forall\ n$$

Then By Uniform Boundness Theorem [1] the sequence $||(P_n)||$ is bounded.

$$\Rightarrow \exists\ K > 0 \text{ such that}$$
$$||P_n|| \le K\ \forall\ n$$
$$\Rightarrow |(J^*v_n,J^*v_n)| = |P_n(J^*v_n)| \le ||P_n||\,||J^*v_n|| \le K||J^*v_n||$$
$$\Rightarrow ||J^*v_n|| \le K\ \forall\ n$$

$\Rightarrow$ the sequence $(||J^*v_n||)$ is bounded which is a contradiction. Hence J^* is bounded.

2. Closed Linear Extension:

2.1 Definition(Closed operator): An Operator $J : D(J) \to H$ is said to be closed if its graph $\upsilon(J) = \{<\varphi,J\varphi>: \varphi \in D(J)\}$ is closed in H × H, where the inner product on H × H is given by

$$(<\varphi,\psi>,<\eta,\mu>) = (\varphi,\eta) + (\psi,\mu)$$

Notation $\tilde{J}$ is said to be an extension of J if

$D(J) \subset D(\tilde{J})$ and $\tilde{J}|D(J) = J$

or

$\upsilon(J) \subset \upsilon(\tilde{J})$

denoted by $J \subset \tilde{J}$

2.2 Definition (Closable): A linear operator J is said to be closable if J has an extension which is closed.

If J is closable then J has a minimal closed extension called closure denoted by $\overline{J}$.[1] **2.3 Proposition:** if J be any closable operator then

$$\upsilon(\overline{J}) = \overline{\upsilon(J)}$$

Proof: Let J be any closable operator $\Rightarrow \overline{J}$ exists.
Let L be any closed extension of J

$$\Rightarrow \upsilon(J) \subset \upsilon(L)$$
$$\Rightarrow \overline{\upsilon(J)} \subset \overline{\upsilon(L)}$$
$$\Rightarrow \overline{\upsilon(J)} \subset \upsilon(L)\ (\because L \text{ is closed})$$

Define an operator P with domain

$$D(P) = \{\mu : <\mu,\eta> \in \overline{\upsilon(J)}\}$$
and
$$P\mu = \eta$$

Claim: <u>P is well defined</u>
Let

$$P\mu = \eta_1 \text{ and } P\mu = \eta_2$$

$$\Rightarrow <\mu,\eta_1>,<\mu,\eta_2> \in \overline{\upsilon(J)}$$
$$\Rightarrow <0,\eta_1-\eta_2> \in \overline{\upsilon(J)}$$
$$\Rightarrow \eta_1 = \eta_2\ (\because \overline{\upsilon(J)} \subset \upsilon(L))$$

Hence well defined. Also

$$\upsilon(\mathcal{P}) = \overline{\upsilon(J)}$$
$$\Rightarrow \mathcal{P} \subset L$$

As L is arbitrary. Therefore P is a minimal closed extension of J

$\Rightarrow P = \overline{J}$. Hence the result.

2.4 Remark A natural question arises here that if we want to take the closure of any linear operator J then we can take the closure of $\upsilon(J)$ in H×H then just find a operator whose graph is equals to $\overline{\upsilon(J)}$.But this is not always possible we constructed an Example 2.5 below.

2.5 Example: Let H be any separable Hilbert Space with countable orthonormal basis $\{\eta_k\}$. Let e be an element of H which is not a finite linear combination of $\{\eta_k\}$. Let

$$e = \sum_{n=1}^{\infty} c_{k_n}\eta_{k_n} \quad (2)$$

Define an operator $J : D(J) \to H$, where
$D(J)$ = set of all the finite linear combination of $\{\eta_k\}$ and e and

$$J(ae + \sum_{i=1}^{N} c_i\eta_k) = ae$$

Clearly J is linear
Claim: $<e,e>,<e,0> \in \overline{\upsilon(J)}$

As

$$e \in D(J) \text{ and } Je = e$$

$$\Rightarrow <\mathfrak{e},\mathfrak{e}> \in \upsilon(J) \subset \overline{\upsilon(J)}$$
$$\Rightarrow <\mathfrak{e},\mathfrak{e}> \in \overline{\upsilon(J)}$$

Also, for each N define

$$\psi_N = \sum_{k=1}^{N} c_{kn}\eta_{kn} \in D(J) \quad \text{and} \quad J\psi_N = 0\ \forall\ N$$

$$\Rightarrow ||<\psi_N,J\psi_N> - <\mathfrak{e},0>||^2 = ||\psi_N - \mathfrak{e}||^2 \to 0 \text{ as } n \to \infty\ (\because \text{By (2)})$$
$$\Rightarrow <\mathfrak{e},0> \in \overline{\upsilon(J)}$$

Suppose $\exists$ an operator L whose graph is $\overline{\upsilon(J)}$
$\Rightarrow Le = e$ and $Le = 0$. Which contradicts the uniqueness.
Hence there is no operator whose graph is $\overline{\upsilon(J)}$

3. Main Results

Theorem 3.1 Let $X : H \times H \to H \times H$ be an operator given by

$$X < \eta,\mu > = < \mu, -\eta >$$

. Then for any linear operator J defined in H with dense domain we have

$$\upsilon(J^*) = [X(\upsilon(J))]^\perp$$

Moreover J^* is closed

Proof: Clearly, X is linear.

Let $< \eta,\mu > \in H \times H$ Then $\exists < -\mu,\eta > \in H \times H$ such that

$$X < -\mu,\eta > = < \eta,\mu >$$

$\therefore X$ is surjective Also

$$\begin{aligned} ||X < \eta,\mu > ||^2 &= || < \mu,-\eta > ||^2 \\ &= ||\mu||^2 + ||\eta||^2 \\ &= || < \eta,\mu > ||^2 \end{aligned}$$

$\therefore X$ is a surjective isometry hence unitary

Claim: $\upsilon(J^*) = [X(\upsilon(J))]^\perp$

Let

$< \eta,\mu > \in [X(\upsilon(J))]^\perp \Leftrightarrow (< \eta,\mu >, X < \varphi, J\varphi >) = 0 \ \forall \ \varphi \in D(J)$

$\Leftrightarrow (< \eta,\mu >, < J\varphi, -\varphi >) = 0 \ \forall \ \varphi \in D(J)$

$\Leftrightarrow < \eta, J\varphi > = < \mu,\varphi > \forall \ \varphi \in D(J)$

$\Leftrightarrow < J\varphi,\eta > = < \varphi,\mu > \forall \ \varphi \in D(J)$

$\Leftrightarrow \eta \in D(J^*), J^*\eta = \mu$

$\Leftrightarrow < \eta,\mu > \in \upsilon(J^*)$

As the graph is closed. Hence J^* is closed

Theorem 3.2 Let J be a closed operator defined in H with dense domain then $D(J^*)$ is dense and $J^{**} = J$

Proof: As

$$\begin{aligned} \overline{\upsilon(J)} &= [\upsilon(J)^\perp]^\perp \\ &= [X^2\upsilon(J)^\perp]^\perp \ (\because X^2 = \text{Identity}) \\ &= [X(X\upsilon(J)^\perp)]^\perp \ (\because X \text{ is unitary}) \\ &= [X\upsilon(J^*)]^\perp \ (\because \text{By Theorem 3.1}) \quad (3) \end{aligned}$$

As J is closed

$$\begin{aligned} \Rightarrow J &= \overline{J} \\ \Rightarrow \upsilon(J) &= \overline{\upsilon(J)} \\ &= [X\upsilon(J^*)]^\perp \end{aligned}$$

Suppose $D(J^*)$ is not dense. Then $\exists\ 0 \neq \eta$ such that

$$\begin{aligned} (\eta,\phi) &= 0 \ \forall \ \phi \in \mathcal{D}(J^*) \\ \Rightarrow (\eta,\phi) + (0, J\phi) &= 0 \ \forall \ \phi \in \mathcal{D}(J^*) \\ \Rightarrow (< \eta, 0 >, < \phi, J\phi >) &= 0 \ \forall \ \phi \in \mathcal{D}(J^*) \\ \Rightarrow < -\eta, 0 > &\in [\mathcal{D}(J^*)]^\perp \\ \Rightarrow < 0, \eta > &\in [X\upsilon(J^*)]^\perp \\ &= \overline{\upsilon(J)} \end{aligned}$$

As $\eta \neq 0$ then there is no linear operator with graph $\upsilon(J)$. which is a contradiction to J is closed.

$\therefore D(J^*)$ is dense $\Rightarrow J^{**}$ exists. Then from Theorem 3.1 and (3) we have

$$\upsilon(J^{**}) = [X\upsilon(J^*)]^\perp = \upsilon(J)$$

$$\Rightarrow J = J^{**}$$

Corollary 3.3:(Closed graph Theorem) A closed operator J defined everywhere on H then J is bounded.

Proof: As J is a linear operator defined everywhere on H. Then J^* exists and By Proposition 1.6 it is also bounded.

Also By Theorem 3.1 we have J^* is closed $\Rightarrow D(J^*)$ is closed [2]. By Theorem 3.2 we have

$$\begin{aligned} \overline{\mathcal{D}(J^*)} &= \mathcal{H} \\ \Rightarrow \mathcal{D}(J^*) &= \mathcal{H} \end{aligned}$$

and $J^{**} = J$

$\Rightarrow J^*$ is defined everywhere on H.

Then again By Proposition 1.6 we have J^{**} is bounded

Hence J is bounded

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