International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 4 Issue 6, September-October 2020 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

The Analytical Nature of the Green's Function in the Vicinity of a Simple Pole

Ghulam Hazrat Aimal Rasa

PhD Doctoral, AL-Farabi Kazakh National University, Almaty, Kazakhstan,
Shaheed Prof. Rabbani Education University, Kabul, Afghanistan

ABSTRACT

It is known that the Green function of a boundary value problem is a meromorphic function of a spectral parameter. When the boundary conditions contain integro-differential terms, then the meromorphism of the Green's function of such a problem can also be proved. In this case, it is possible to write out the structure of the residue at the singular points of the Green's function of the boundary value problem with integro-differential perturbations. An analysis of the structure of the residue allows us to state that the corresponding functions of the original operator are sufficiently smooth functions. Surprisingly, the adjoint operator can have non-smooth eigenfunctions. The degree of non-smoothness of the eigenfunction of the adjoint operator to an operator with integro-differential boundary conditions is clarified. It is indicated that even those conjugations to multipoint boundary value problems have non-smooth eigenfunctions.

KEYWORDS: *boundary conditions, Green's function Operator, strongly regular boundary conditions, Riesz basis, asymptotics, Resolution, boundary functions, simple zero, adjoint operator*

How to cite this paper: Ghulam Hazrat Aimal Rasa "The Analytical Nature of the Green's Function in the Vicinity of a Simple Pole" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-4 | Issue-6, October 2020, pp.1750-1753, URL: www.ijtsrd.com/papers/ijtsrd33696.pdf

IJTSRD33696

1. INTRODUCTION

Let be $0<x<1$ and a differential expression is given

$$L(\mathrm{y}) = \mathrm{y}^{(n)}(x) + \sum_{k=0}^{n-1} p_k(\mathrm{x})\,\mathrm{y}^{(\mathrm{k})}(\mathrm{x}), \qquad 0<\mathrm{x}<1$$

With smooth coefficients $p_k \in C^k[0,1], \quad \mathrm{k}=0,1,\dots,\mathrm{n}-1$

All further arguments are shown for case n, however, the presented results (with their corresponding modification) are satisfied for arbitrary n.

Find a set of numbers $\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2,\dots,\alpha_{n-1},\beta_{n-1},\gamma_{n-1},\alpha_n,\beta_n,\gamma_n$ such that $\theta_0 \neq 0$at $\theta_1 \neq 0$, where

$$\theta_0+\theta_1 s = \begin{vmatrix} \alpha_1\omega_1^{\gamma_1} & \cdots & \alpha_1\omega_{\mu-1}^{\gamma_1} & (\alpha_1+s\beta_1)\omega_\mu^{\gamma_1} & \beta_1\omega_{\mu+1}^{\gamma_1} & \cdots & \beta_1\omega_n^{\gamma_1} \\ \alpha_2\omega_1^{\gamma_2} & \cdots & \alpha_2\omega_{\mu-1}^{\gamma_2} & (\alpha_2+s\beta_2)\omega_\mu^{\gamma_2} & \beta_2\omega_{\mu+1}^{\gamma_2} & \cdots & \beta_2\omega_n^{\gamma_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_n\omega_n^{\gamma_n} & \cdots & \alpha_n\omega_{\mu-1}^{\gamma_n} & (\alpha_n+s\beta_n)\omega_\mu^{\gamma_n} & \beta_n\omega_{\mu+1}^{\gamma_n} & \cdots & \beta_n\omega_\mu^{\gamma_n} \end{vmatrix}.$$

From the results of [1], [2] it follows that the set of boundary conditions

$$\begin{aligned} U_1(y) &= \alpha_1 y^{(\gamma_1)}(0)+\beta_1 y^{(\gamma_1)}(1)=0, \\ U_2(y) &= \alpha_2 y^{(\gamma_2)}(0)+\beta_2 y^{(\gamma_2)}(1)=0 \\ &\dots \qquad \dots \\ U_n(y) &= \alpha_n y^{(\gamma_n)}(0)+\beta_n y^{(\gamma_n)}(1)=0 \end{aligned} \qquad (1)$$

represent intensely regular boundary conditions. Therefore, the system of eigen functions of the eigenvalue problem

$$L(y)=\lambda y, \quad 0<x<1 \qquad (2)$$

With boundary conditions (1) form a Riesz basis in the functional space $L_2(0,1)$.

According to the monograph [3], the asymptotics of one series of eigenvalues of the boundary value problem (1), (2) has the following form

$$\lambda_k^{'} = (-2k\pi i)^3 \left[1 - \frac{3\ln_0 \xi^{(1)}}{2k\pi i} + O(\frac{1}{k^2}) \right]$$

Another series of eigenvalues of the boundary value problem (1), (2) has a similar form.

From the results of [4], it follows that the following boundary conditions

$$V_j(y) \equiv U_j(y) + \sum_{s=0}^{\gamma_j} \int_0^1 y^{(s)}(t)\rho_{s_j}(t)dt = 0, \; j=1,\dots,n \qquad (3)$$

also represent reinforced- edge conditions. Therefore, the results of [1], [2] remain valid, so that the system of eigenfunctions and associated functions of the operator with boundary conditions (3) form a Riesz basis in the function space $L_2(0,1)$.

For further purposes, boundary conditions (3) are conveniently rewritten in the canonical form proposed in [5].

The equation $L(y)=f(x), \quad 0<x<1$ with integral-differential conditions of the form

$$V_j(y)\equiv U_j(y)-\int_0^1 L(y)\overline{\sigma_j(x)}dx=0, \qquad j=1,2,\dots,n \quad (4)$$

for an arbitrary set of boundary functions

$$\sigma_1\in L_2(0,1),\sigma_2\in L_2(0,1),\dots,\sigma_n\in L_2(0,1).$$

Has the only solution $y(x)$ for any f of $L_2(0,1)$ moreover, the estimate $\|y\|_{L_2(0,1)}\le c\|f(x)\|_{L_2(0,1)}$. The converse is also true. If the equation $L(y)=f(x)$, $0<x<1$ with some additional linear conditional for any f of $L_2(0,1)$ has the only solution requiring $\|y\|_{L_2(0,1)}\le c\|f(x)\|_{L_2(0,1)}$ then there is such a set of boundary functions $\sigma_1\in L_2(0,1),\sigma_2\in L_2(0,1),\dots,\sigma_n\in L_2(0,1)$, that additional conditions will be equivalent to conditions (3).

According to the above theorem, conditions (3) are equivalent to conditions (4) for some $\sigma_1\in L_2(0,1),\sigma_2\in L_2(0,1),\dots,\sigma_n\in L_2(0,1)$ Details of the calculation of boundary functions $\sigma_1,\sigma_2,\dots,\sigma_n$ by function $\{\rho_{sj}(t)\}$ can be found in [6].

Objectives of this research

The main purpose of this research paper is to obtain the analytical nature of Green's functions in the vicinity of a simple pole.

Methodology:

A descriptive research project to focus and identify the effect of differential equations on the analytical nature of the Green's functions in the vicinity of a simple pole. Books, journals, and websites have been used to advance and complete this research.

2. Literature review

The article is devoted to the construction of the Green function of the boundary value problem for a differential equation with strongly regular boundary conditions in the vicinity of the pole. Questions such as the construction of the Green function and expansion in eigenfunctions for differential operators with strongly regular boundary conditions are poorly understood. To study the analytical nature of the Green's function, we develop the ideas of [Kanguzhin, The residue and spectral decompositions of a differential operator on a star graph, VestnikKazNU, 2018], [Keldysh]. We note the papers [9], [10] devoted to the expansion in eigenfunctions of differential self-adjoint and non-self-adjoint operators.

The inverse problems for differential operators with regularly boundary conditions for the second order were studied in [14]. In [17], spectral problems for an odd-order differential operator are considered. In [13], [15], [18], [19], [20] some questions of spectral analysis of inverse problems for differential operators are presented.

And in this paper, we derive a formula for expanding the Green's function into eigenfunctions of a third-order differential operator with strongly regular boundary conditions.

3. Green function of an unperturbation boundary value problem

Consideration of the boundary value eigenvalue problem

$$L(y)=f(x), \quad 0<x<1$$

$$U_1(y)=0,U_2(y)=0,\dots,U_n(y)=0$$

Operator resolution L has the form

$$(L_0-\lambda I)^{-1}f=\int_0^1 G_0(x,t,\lambda)f(t)dt$$

where

$$G_0(x,t,\lambda)=(-1)^n\frac{\begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) & \cdots & y_n(x,\lambda) & g(x,t)\\ U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) & U_1(g)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) & U_n(g)\end{vmatrix}}{\Delta_0(\lambda)}$$

-is the Green's function of the operator.

Here at $x>t$ function $g(x,t)$ has the following form:

$$g(x,t)=\begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & \cdots & y_n^{(1)}(t)\\ \vdots & \vdots & \ddots & \vdots\\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t)\\ y_1(x) & y_2(x) & \cdots & y_n(x)\end{vmatrix},$$

If $x\le t$, then $g(x,t)=0$.

$$\Delta_0(\lambda)=\begin{vmatrix} U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n)\\ U_2(y_1) & U_2(y_2) & \cdots & U_2(y_n)\\ \vdots & \vdots & \ddots & \vdots\\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n)\end{vmatrix}$$

3.1. Perturbed boundary value problem and its Green function

Consider the boundary value eigenvalue problem

$$L(y)=f(x), \quad 0<x<1$$

$$V_1(y)\equiv U_1(y)-\int_0^1 L(y)\overline{\sigma_1(x)}dx=0$$

$$U_2(y)=0,U_3(y)=0,\dots,U_n(y)=0$$

Operator Resolution L has the form

$$(L-\lambda I)^{-1}f(x)=\int_0^1 G(x,t,\lambda)f(t)dt, \qquad 0<x<1$$

Where

$$G(x,t,\lambda)=\frac{\begin{vmatrix}\kappa_1(x,\lambda) & G_0(x,t,\lambda)\\ V_1(\kappa_1) & V_1(G_0)\end{vmatrix}}{V_1(\kappa_1)}=G_0(x,t,\lambda)-\frac{\kappa_1(x,\lambda)V_1(G_0)}{V_1(\kappa_1)}$$

- Green's function of the operator L.

Here $\kappa_1(x,\lambda)$ - solution of a homogeneous equation

$$L(\kappa_1)=\lambda\kappa_1 \quad , \qquad 0<x<1,$$

With heterogeneous boundary conditions

$$U_1(\kappa_1)=1, U_2(\kappa_1)=0, ..., U_n(\kappa_1)=0.$$

3.2. The main part of the Laurent series expansion of the Green function

In this section, the main part of the Laurent expansion of the Green function of the perturbed operator in a neighborhood of a simple eigenvalue is calculated. In our case, the zero of the function $V_1(\kappa_1)$ are the poles of the resolvent $(L-\lambda I)^{-1}$. Let be λ_0- simple zero function $V_1(\kappa_1)=0$ and $\left.\frac{d}{d\lambda}V_1(\kappa)\right|_{\lambda_0}\neq 0$. Then the expansion in the Laurent series in the vicinity of a simple pole takes the following form

$$G(x,t,\lambda)=-\frac{\underset{\lambda_0}{res}\,G(x,t,\lambda)}{\lambda-\lambda_0}+Right\ part,$$

where

$$\underset{\lambda_0}{res}\,G(x,t,\lambda)=-\frac{\kappa_1(x,\lambda)V_1(G_0)}{\left.\frac{d}{d\lambda}V_1(\kappa_1)\right|_{\lambda=\lambda_0}} \qquad (5)$$

By the Keldysh theorem [7] it is known that

$$G(x,t,\lambda)=-\frac{u_0(x)\overline{v_0(t)}}{\lambda-\lambda_0}+h_0(x,t,\lambda), \qquad (6)$$

where $h_0(x,t,\lambda)-$regular in a neighborhood of a point λ_0. Here $u_0(x)-$own operator function L, and $\overline{v_0(t)}$ - eigenfunction of the adjoint operator L^* to the operator L, that is to say

$$L(u_0)=\lambda_0 u_0, \qquad u_0\in D(L),$$
$$L^*(v_0)=\overline{\lambda_0}v_0, \qquad v_0\in D(L^*).$$

comparing relations (5) and (6), we obtain the equalities

$$u_0(x)=\kappa_1(x,\lambda_0), \qquad v_0(t)=\frac{V_1(G_0)}{\left.\frac{d}{d\lambda}V_1(\kappa_1)\right|_{\lambda=\lambda_0}}$$

Theorem 1. If λ_0- simple eigenvalue of the operator L, then the eigenfunction of the operator L has the following form

$$k_1(x,\lambda_0)=\frac{\begin{vmatrix}y_1(x,\lambda) & y_2(x,\lambda) & \cdots & y_n(x,\lambda)\\ U_2(y_1) & U_2(y_2) & \cdots & U_2(y_2)\\ \vdots & \vdots & \ddots & \vdots\\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n)\end{vmatrix}}{\Delta_0(\lambda)}$$

That is, it satisfies the following perturbed boundary value problem

$$L(\kappa_1(x,\lambda_0))=\lambda_0\kappa_1(x,\lambda_0),$$
$$V_1(\kappa_1(x,\lambda_0))=U_1(\kappa_1(x,\lambda_0))-\int_0^1 L(\kappa_1(x,\lambda_0))\overline{\sigma_1(x)}dx=0,$$
$$U_2(\kappa_1(x,\lambda_0))=\cdots=U_n(\kappa_1(x,\lambda_0))=0$$

Theorem 2. Let be λ_0- simple zero function $V_1(\kappa_1)$. Then the Green's function of the operator L in the vicinity of a simple pole has the representation

$$G(x,t,\lambda)=\frac{\dfrac{\kappa_1(x,\lambda_0)V_1(G_0)}{\left.\frac{d}{d\lambda}V_1(\kappa_1)\right|_{\lambda=\lambda_0}}}{\lambda-\lambda_0}+G_0(x,t,\lambda_0)$$

Here $\kappa_1(x,\lambda_0)$ - own operator function L, $V_1(G_0)$ - eigenfunction of the adjoint operator L^* to the operator L. Moreover, it is determined by the following formula

$$V_1(G_0)=\sigma_1(t)+\overline{\lambda_0}\int_t^1\overline{g(x,t,\lambda)}\sigma_1(x)dx+$$

$$+\frac{\overline{\lambda_0}}{\Delta_0(\lambda_0)}\begin{vmatrix} y_1(t) & y_2(t) & y_n(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_n^{(1)}(t)\\ \vdots & \vdots & \vdots\\ y_1^{(k-2)}(t) & y_2^{(k-2)}(t) & y_n^{(k-2)}(t)\\ \int_0^1\sum_{k=1}^{n-1}\beta_{k+1}y_1^{(k)}(1)A_{n+1,k+2}\overline{\sigma_1(x)}dx & \int_0^1\sum_{k=1}^{n-1}\beta_{k+1}y_2^{(k)}(1)A_{n+1,k+2}\overline{\sigma_1(x)}dx & \int_0^1\sum_{k=1}^{n-1}\beta_{k+1}y_n^{(k)}(1)A_{n+1,k+2}\overline{\sigma_1(x)}dx\end{vmatrix}$$

4. Conclusion

In conclusion, we note that the eigenfunction of the original operator is a smooth function. At the same time, the eigenfunction of the adjoint operator cannot be smooth and the degree of its smoothness depends on the smoothness of the boundary perturbation. These conclusions follow from the representation of the eigenfunctions of the data in Theorem 2. In particular, even those conjugations to multipoint boundary value problems possess non-smooth eigenfunctions. In this theorem 2, we give results concerning the perturbation of only one boundary condition. Similar results are obtained when all three boundary conditions are perturbed.

References

[1] Михайлов В.П. О базисах Рисса // ДАН СССР. – 2018. – № 5 (144). – С.981-984.

[2] Кесельман Г. М. Обезусловной сходимости разложений по собственным функциям некоторых дифференциальныхоператоров // Изв. Вузов СССР, Математика.–2017.–№2.–С.82-93.

[3] Наймарк М. А. Линейные дифференциальные операторы. – М., 2019. –528с.

[4] Шкаликов А. А. О базисности собственных функций обыкновенных дифференциальных операторов с интегральными краевыми условиями // Вестн.МГУ.Сер.Мат.Мех.–2019.–№6.–С.12-21.

[5] Кокебаев Б. К., Отелбаев М., Шыныбеков А. Н. К вопросам расширений и сужений операторов // Докл. АН СССР.– 1983. – № 6 (271). – С.1307-1313.

[6] КангужинБ. Е., Даирбаева Г., Мадибайулы Ж. Идентификация граничных условий дифференциального оператора// Вестник КазНУ. Серия математика, механика, информатика. – 2019. – № 3 (103). – С.13-18

[7] ДезинА. А. Дифференциально-операторныеуравнения. Метод модельных операторов в теории граничных задач// Тр. МИАН. М., Наука. МАИК «Наука/Интерпериодика».–2000.–№229.–С.3-175

[8] Левитан Б. М. Обратная задача для оператора Штурма-Лиувилля в случае конечно-зонных и бесконечно –зонныхпотенциалов // Труды Моск. Матем.об-во.-МГУ.-М.–1982.–С.3-36.

[9] Березанский Ю.М. Разложение по собственным функциям. – М.-Л, 1950.

[10] БерезанскийЮ. М. Разложение по собственным функциям самосопряженныхоператоров.–Киев: НауковаДумка, 1965.–798с.

[11] Марченко В. А. Операторы Штурма-Луивилля и их приложения. – Киев: НауковаДумка, 1977.–329с.

[12] ато Т. Теория возмущенний линейных операторов. – М.: Мир, 1972. – 740с.

[13] Лейбензон З.Л. Обратная задача спектрального анализа обыкновенных дифференциальныхоператоров высших порядков /З.Л. Лейбензон //Труды Москов. мат. об-ва. – 1966. – Т. 15. – С.70-144.

[14] Юрко В.А. Обратная задача для дифференциальных операторов второго порядка с регулярными краевыми условиями / В.А. Юрко // Мат. заметки.–1975.–Т.18.–№4.–С.569-576.

[15] Садовничий В. А. О связи между спектром дифференциального оператора с симметричными коэффицентами икраевыми условиями / В. А. Садовничий, Б. Е. Кангужин // ДАН СССР. – 1982. – Т. 267. – №2. – С.310-313.

[16] Шкаликов А. А. О базисности собственных функций обыкновенных дифференциальных операторов с интегральными краевыми условиями / А.А. Шкаликов // Вестник МГУ. Сер. Мат. Мех. – 1982. – № 6. – С.12-21

[17] Станкевич М. Об одной обратной задаче спектрального анализа для обыкнове иного дифференциального операторачетного порядка / М. Станкевич // Вестник МГУ. Сер. Мат. Мех. – 1981. – № 4. – С.24-28.

[18] Ахтямов А. М. Обобщения теоремы единственности Борга на случай неразделенныхграничных условий/А.М. Ахтямов В.А. Садовничий, Я. Т. Султанаев // Евразийская математика. – 2012. – Т. 3. – №4. – С.10-22.