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Generalized Derivations in Prime near Rings

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ABSTRACT

Let NP be a zero symmetric prime near ring with multiplicative centre Z. Let f: NP → NP be a generalized derivation defined on NP. We prove that "If f ≠ 0 generalized derivation on NP for which (a) f(NP) ⊆ Z (b) [f(x),f(y)] = 0 ∀x,y∈ NP " Also if NP is 2-torsion free then NP is commutative ring, from which Herstein [2] Theorem comes out as a corollary.

KEYWORDS: *Prime near rings, Generalized Derivation, Commutative ring, 2-Torsion free.*

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1. INTRODUCTION

In this paper NP will denote a zero-symmetric near-ring with multiplicative centre Z. A generalized derivation on NP is defined to be an additive endomorphism satisfying

f(xy) = f(x)y + xD(y) ∀x,y∈ N

where D is the ordinary derivation defined on NP.

For x,y∈ NP, the symbol [x,y] will denote the commutator xy−yx, while the symbol (x,y) will denote the additive-group commutator x+y−x−y.

The generalized derivation f will be called commuting if [x,f(x)] = 0 ∀ x ∈ NP. Finally, NP will be called prime if a,b∈ N and aNb = {0} implies that a = 0 or b = 0.

(Note that this definition implies the usual definition of prime near-ring. It does not seem to be known whether the twodefinition are equivalent.)

We have proved (1) If f ≠ 0 generalized derivation on prime near ring N for which

A. f(NP) ⊆ Z
B. [f(x),f(y)] = 0 ∀x,y∈ NP

Also if NP is 2-torsion free then NP is commutative ring, from which Herstein [2] Theorem comes out as a corollary.

2. Preliminary results

We begin with three quite general and useful lemmas to proving Theorems in Prime near Rings.

Lemma 3.1 Let f be an arbitrary generalized derivation on the near-ring N. Then N satisfies the following Partial Distributive Law:

(f(a)b + aD(b))c = f(a)bc + aD(b)c ∀a,b,c∈ NP

Lemma 3.2 If f be a generalized derivation on N and suppose that u ∈ N is not a zero divisor. then (u,x) is constant for every x ∈ N.

Lemma 3.3 Let N have no non-zero divisors of zero if N admits a generalized
derivation f. Then (N,+) is abelian.

3. Prime near-rings:

We have taken NP be the prime near-ring.

Lemma 4.1 Let NP be a Prime near-ring (i) If z ∈ Z −{0} then z is not zero divisor (ii) If Z contains a non-zero element z for which z + z ∈ Z then (NP,+) is abelian.

Proof

A. If z ∈ Z −{0} and zx = 0. Then zNPx = 0. Hence x = 0.
B. Let z ∈ Z−{0}be an element such that z+z∈ Z and let x,y∈ NP. Since z + z is distributive

⇒ (x + y)(z + z) = x(z + z) + y(z + z)
= xz + xz + yz + yz
= (x + x + y + y)z

On the other hand
(x + y)(z + z) = (x + y)z + (x + y)z
= xz + yz + xz + yz
= (x + y + x + y)z

Then x + x + y + y = x + y + x + y.
Hence x + y = y + x
Hence (NP,+) is abelian.

Lemma 4.2 Let NP be a Prime near-ring

A. Let f be a non-zero generalized derivation on NP. Then xf(NP) = {0} ⇒ x = 0, and f(NP)x = {0} implies x = 0.
B. If NP is 2-torsion free and f is a generalized derivation on NP s.t. f2 = 0 then f = 0

Proof (i) Let xf(NP) = {0} and let r,s be arbitrary elements of NP. Then

xf(rs) = 0 ⇒ x(f(r)s + rD(s)) = 0
⇒xf(r)s =0 + xrD(s) = 0
⇒xrD(s) = 0

Then xND(NP) = {0} Since D(NP) ≠ 0 ⇒ x = 0

A similar argument works if f(NP)x = {0} (Since Lemma 3.1 gives distributivity to carry it through).
Let x, y ∈ NP we have
0 = f2(xy)
⇒ 0 = f (f(x)y + xD(y))
= f (f(x))y + f(x)D(y) + f(x)D(y) + xD(D(y))
= f2(x)y + f(x)D(y) + f(x)D(y) + xD2(y)
⇒ 0 = 2f(x)D(y)
⇒ f(x)D(y) = 0
since NP is 2 torsion-free ⇒ f(x)D(NP) = {0} for each x ∈ N and (i) gives f = 0.

Theorem 4.3 Let NP be a Prime near-ring. Let f be a non zero generalized derivation for which f(NP) ⊆ Z, then (NP,+) is abelian. Moreover, if NP is 2-torsion-free then NP is a commutative ring.

Proof Let v be an arbitrary constant, and let x be a non constant. Then
f(xv) = f(x)v + xD(v)
= f(x)v ∈ Z

Since f(x) ∈ Z −{0}, it follows easily that v ∈ Z. Since v + v is constant for all constants v, it follows from Lemma 2.1(ii) that (NP,+) is abelian, provided that there exists a non-zero constant. Let NP is 2-torsion free near-ring. To prove that NP is commutative.

By Lemma 3.1
(f(a)b + aD(b))c = f(x)bc+aD(b)c ∀a,b,c∈ N and using the fact that f(ab) ∈ Z, we get

cf(a)b + caD(b) = f(a)bc + aD(b)c

Since (NP,+) is abelian and f(NP) ⊆ Z. This equation rearrange to yield f(a)[b,c] = D(b)[c,a], ∀a,b,c∈ NP Now suppose that NP is not commutative we choose b,c∈ NP with [b,c] ≠ 0 and letting a = f(x), we get

f2(x)[b,c] = 0 ∀ x ∈ NP;
⇒ we conclude that

f2(x) = 0 ∀ x ∈ NP By Lemma 4.2(ii) f = 0 which is a contradiction ∵ f 6= 0. Hence our supposition is wrong. So NP is commutative.

Theorem 4.4 Let NP be a Prime near-ring admitting a non zero generalized derivation f such that [f(x),f(y)] = 0 ∀x,y∈ NP. Then (NP,+) is abelian and if NP is 2-torsion free as well. Then NP is a commutative ring.

Proof By Lemma 4.1(ii), if z and z + z commute elementwise with f(NP). Then z f(v) = 0 for all additive commutators v. Thus putting z = f(x), we get

f(x)f(v) = 0 ∀ x ∈ NP
⇒ f(c) = 0

(By Lemma 4.2(i)) Since wv is also an additive commutator for any w ∈ NP, we have
f(wv) = 0 ⇒ f(w)v = 0

By Lemma 3.3
v = 0.

Hence (NP,+) is an abelian. Assume now that NP is 2-torsion free. By partial distributive law

f(f(x)y)f(z) = (f(f(x)y + f(x)D(y))f(z) ∀x,y,z∈ NP
= f2(f(x))y f(z) + f(x)D(y)f(z)
⇒ f2(f(x))yf(z) = −f(f(x)y)f(z)−f(x)D(y)f(z)
= f(z)(f(f(x)y−f(x)D(y))
= f(z)(f(f(x)y)
= f(z)f2(x)y
= f2(x)f(z)y
⇒ f2(x)(yf(z)−f(z)y) = 0 ∀x,y,z∈ NP

Replacing y by yt we get
f2(x)(ytf(z)−f(z)yt) = 0
⇒ f2(x)ytf(z) = f2(x)f(z)yt
= f2(x)yf(z)t∀x,y,z,t∈ NP
⇒ f2(x)NP [t,f(z)] = {0} ∀x,t,z∈ NP
⇒ either f2 = 0 or f(NP) ⊆ Z (∵ NP is Prime)

If f2 = 0 ⇒ By Lemma 4.2 (ii) f = 0
which is a contradiction. (∵ f ≠0)
⇒ f2 6= 0
⇒ f(NP) ⊆ Z.

By Theorem 4.3, NP is a commutative ring.
Hence proved.

Corollary 4.4.1 Replacing f by D we get Herstein [2] Theorem

Conclusion

In this paper we proved that "If f ≠ 0 generalized derivation on NP for which (a) f(NP) ⊆ Z (b) [f(x),f(y)] = 0 ∀x,y∈ NP " and we also showed that if NP is 2-torsion free then NP is commutative ring, from which Herstein [2] Theorem comes out as a corollary.

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