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Strum - Liouville Problems in Eigenvalues and Eigenfunctions

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ABSTRACT

This paper we discusses with Strum-Liouville problem of eigenvalues and eigenfunctions, within the standard equation $\frac{d}{dx}\left(r\frac{d\phi(x)}{dx}\right)+(q+\lambda p)\phi(x)=0$ where p,q and r are given functions of the independent variable x is an interval $a \le x \le b$. The λ is a parameter and $\phi(x)$ is the dependent variable. The method of separation of variable applied to second order liner partial differential equations, the equation is known because the Strum-Liouville differential equation. Which appear in the overall theory of eigenvalues and eigenfunctions and eigenfunctions expansions is one of the deepest and richest parts of recent mathematics. These problems are associate with work of J.C.F strum and J.Liouville.

KEYWORDS: *Strum-Liouville problem of eigenvalues and eigenfunctions, eigenvalues and eigenfunctions, orthogonal, weight functions*

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INTRODUCTION

The method of separation of variables utilized within the solutions of boundary value problems of mathematical physics frequently gives rise so called Strum-Liouville eigenvalue problems, our attention to small but significant fragment of the theory of Strum-Liouville problems and their solutions. The character of the any spectrum of particular problem will be determined by actually finding the eiganvalue. We shall establish the orthogonallity of eigenfunctions corresponding to distinct eigenvalues, orthogonal set with regard to weight functions that not piecewise continuous.

1.1. Eigenvalue, Eigenfunction

The value of λ, for that the strum-Liouville system (1.1) feature a nontrivial solution are called the eigenvalues, and thus corresponding solution are called the eignfunctions.

1.2. Strum-Liouville system

The solution of partial differential equation by the method of separation of variables.

Under separable conditions we'll transform the second order homogeneous partial differential equation into ordinary differential equation.

$$a_1x'' + a_2x' + (a_3+\lambda)x = 0 \qquad \ldots(1.1)$$

$$b_1y'' + b_2y' + (b_3+\lambda)y = 0 \qquad \ldots(1.2)$$

Which are of the form

$$a_1(x)\frac{d^2y}{dx^2} + a_2(x)\frac{dy}{dx} + \left[a_3(x)+\lambda\right]y = 0 \ldots(1.3)$$

i.e. two strum Liouville problems we introduce

$$r(x) = \exp\left[\int^x \frac{a_2(t)}{a_1(t)}\right]dt$$

$$q(x) = \frac{a_3(x)}{a_1(x)}r(x),\ p(x) = \frac{r(x)}{a_1(x)} \qquad \ldots(1.4)$$

Equation (1.3) we obtain

$$\frac{d}{dx}\left(r\frac{dy}{dx}\right) + (q+\lambda p)y = 0 \qquad \ldots(1.5)$$

This is mentioned because the strum-Liouville equation. In terms of the operator

$$L \equiv \frac{d}{dx}\left(r\frac{d}{dx}\right) + q$$

Equation (1.5) are often written as

$$L(y) + \lambda p(x)y = 0 \qquad \ldots(1.6)$$

Where λ is a parameter independent of x, and p, q and r are real-valued functions of x.

To ensure the existence of solution, we let p and q be continuous and r be continuously differential during a closed finite interval (a, b).

1.3. Theorem

Let the coefficients p, q and r within the Strum-Liouville system be continuous in (a, b).Let the eigenfunctions ϕ_j and ϕ_k, corresponding to λ_j and λ_k, be continuously differentiable, then ϕ_j and ϕ_k are orthogonal with regard to the load function p(x) in (a, b).

Proof:

Since ϕ_j corresponding to λ_j satisfies the strum-Liouville equation, we've

$$\frac{d}{dx}\left(r\phi_j'\right)+\left(q+\lambda_j p\right)\phi_j=0 \qquad ...(1.7)$$

For the same reason

$$\frac{d}{dx}\left(r\phi_k'\right)+\left(q+\lambda_k p\right)\phi_k=0 \qquad ...(1.8)$$

Multiplying equation (1.7) by ϕ_k and equation (1.8) by ϕ_j, and subtracting we obtain

$$\left(\lambda_j-\lambda_k\right)p\phi_j\phi_k=\phi_j\frac{d}{dx}\left(r\phi_k'\right)-\phi_k\frac{d}{dx}\left(r\phi_j'\right)$$

$$=\frac{d}{dx}\left(r\phi_k'\right)\phi_j-\frac{d}{dx}\left(r\phi_j'\right)\phi_k$$

and integrating yields

$$\left(\lambda_j-\lambda_k\right)\int_a^b p\phi_j\phi_k dx=\left[r\left(\phi_j\phi_k'-\phi_j'\phi_k\right)\right]_a^b$$

$$=r(b)\left[\phi_j(b)\phi_k'(b)-\phi_j'(b)\phi_k(b)\right]$$

$$-r(a)\left[\phi_j(a)\phi_k'(a)-\phi_j'(a)\phi_k(a)\right] \qquad ...(1.9)$$

The right side of which is knows as the boundary term of the Strum-Liouville system.

The end condition for the eigenfunctions ϕ_j and ϕ_k are

$$b_1\phi_j(b)+b_2\phi_j'(b)=0$$

$$b_1\phi_k(b)+b_2\phi_k'(b)=0$$

If $b_2\neq 0$, we multiple the first condition by $\phi_k(b)$ and the second condition by $\phi_j(b)$, and subtract to obtain

$$\phi_j(b)\phi_k'(b)-\phi_j'(b)\phi_k(b)=0 \qquad ...(1.10)$$

In a similar manner, if $a_2\neq 0$, we obtain

$$\phi_j(a)\phi_k'(a)-\phi_j'(a)\phi_k(a)=0 \qquad ...(1.11)$$

We see by virtue of (1.10) and (1.11) that

$$\left(\lambda_j-\lambda_k\right)\int_a^b p\phi_j\phi_k dx=0 \qquad ...(1.12)$$

if λ_j *and* λ_k are district eigenvalues, then

$$\int_a^b p\phi_j\phi_k dx=0 \qquad ...(1.13)$$

1.4. Theorem

All the eigenvalues are regular strum-Liouville system with p(x) > 0 are real.

Proof:

Suppose that there is a complicated eigenvalue

$\lambda_j=\alpha+i\beta$

With eigenfunction $\phi_j=u+iv$

Then, because the coefficient of the equation are real.

The complex conjugate of the eienvalue is also eigenvalue.

Thus,there exists an eigenfunction

$\phi_k=u-iv$

Corresponding to the eigenvalue $\lambda_k=\alpha-i\beta$

By using the relation (1.12), we have

$$2i\beta\int_a^b p\left(u^2+v^2\right)dx=0$$

This implies that β must vanish for p>0 and hence the eigenvalues are real.

1.5. Theorem

If $\phi_1(x)$ and $\phi_2(x)$ are any two solutions of $L(y)+\lambda p(y)=0$ on (a, b), then $r(x)w(x:\phi_1,\phi_2)=constant$

Where w is the wronskian.

Proof:

Since ϕ_1 *and* ϕ_2 are solution of

$L(y)+\lambda p(y)=0$

We have

$$\frac{d}{dx}\left(r\frac{d\phi_1}{dx}\right)+(q+\lambda p)\phi_1=0 \qquad ...(1.14)$$

$$\frac{d}{dx}\left(r\frac{d\phi_2}{dx}\right)+(q+\lambda p)\phi_2=0 \qquad ..(1.15)$$

Multiplying the equation (1.14) by ϕ_2 and the equation (1.15) by ϕ_1 and subtracting, we obtain

$$\phi_1\frac{d}{dx}\left(r\frac{d\phi_2}{dx}\right)-\phi_2\frac{d}{dx}\left(r\frac{d\phi_1}{dx}\right)=0$$

Integrating this equation from a to x, we obtain

$$r(x)\left[\phi_1(x)\phi_2'(x)-\phi_1'(x)\phi_2(x)\right]$$

$$=r(a)\left[\phi_1(a)\phi_2'(a)-\phi_1'(a)\phi_2(a)\right] \qquad ...(1.16)$$

= constant

This is called Abel's formula.

1.6. Theorem

An eigenfunction of normal strum-Liouville system is unique except for a constant factor.

Proof:
Let $\phi_1(x)$ and $\phi_2(x)$ are eigenfunctions corresponding to an eigenvalue λ.

The according to Abel's formula we have
$r(x)\,w(x:\phi_1,\phi_2)=constant\; r(x)>0$
Where w is the wronskian.

Thus, if w vanishes at a point in (a, b) it must vanish for all $x\in(a,b)$

Since ϕ_1 *and* ϕ_2 satisfy the end condition at x = a, we have

$$a_1\phi_1(a)+a_2\phi_1'(a)=0$$

$$a_1\phi_2(a)+a_2\phi_2'(a)=0$$

Since a_1 and a_2 are not both zero.

We have $$\begin{vmatrix}\phi_1(a) & \phi_1'(a)\\ \phi_2(a) & \phi_2'(a)\end{vmatrix}=w(a:\phi_1,\phi_2)=0$$

Therefore $w(x:\phi_1,\phi_2)=0$ for all x at (a, b), which is a sufficient condition for the linear dependence of two functions ϕ_1 *and* ϕ_2. Hence $\phi_1(x)$ differs from $\phi_2(x)$ only by a constant factor.

1.7. Theorem
The eigenvalues of a strum-Liouville system are real.

Proof:
Let the strum-Liouville system be the second order differential equation

$$\frac{d}{dx}(r\,y')+(q+\lambda p)y=0 \qquad ...(1.17)$$

In the interval $a\le x\le b$ where p, q, r are real functions and $p(x)\ge 0$, together with the boundary conditions.

$$a_1y(a)+a_2y'(a)=0 \qquad ...(1.18)$$

$$b_1y(b)+b_2y'(b)=0 \qquad ...(1.19)$$

We assume that a1, a2, b1, b2 is real while λ and y may be complex.

Taking the complex conjugate (1.17), (1.18) and (1.19) we have,

$$\frac{d}{dx}\left(\bar{r}\,\bar{y}'\right)+(q+\bar{\lambda}p)\bar{y}=0 \qquad ...(1.20)$$

$$a_1\bar{y}(a)+a_2\bar{y}'(a)=0 \qquad ...(1.21)$$

$$b_1\bar{y}(b)+b_2\bar{y}'(b)=0 \qquad ...(1.22)$$

(1.17) x $\bar{y}$ - (1.20) x y gives

$$\bar{y}\frac{d}{dx}(r\,y')-y\frac{d}{dx}\left(r\bar{y}'\right)+py\bar{y}(\lambda-\bar{\lambda}) = 0$$

$$\frac{d}{dx}r\left(\bar{y}y'-y\bar{y}'\right)=p\,y\,\bar{y}(\bar{\lambda}-\lambda) \qquad ...(1.23)$$

Integrating both sides of (1.23) with respect to x from a to b, we get

$$\int_a^b p' y_1^2(\bar{\lambda}-\lambda)dx=r\left(\bar{y}y'-y\bar{y}'\right)_a^b$$

$$=r(b)\left[\bar{y}(b)y'(b)-y(b)\bar{y}'(b)\right]$$

$$-r(a)\left[\bar{y}(a)y'(a)-y(a)\bar{y}'(a)\right] \qquad ...(1.24)$$

Now (1.21) x $y'(a)$ - (1.18) x $\bar{y}'(a)$ gives

$$a_1\left[\bar{y}(a)y'(a)-y(a)\bar{y}'(a)\right]=0$$

$$As\; a_1\neq 0,\bar{y}(a)y'(a)-y(a)\bar{y}'(a)=0$$

Similarly $\bar{y}(b)y'(b)-y(b)\bar{y}'(b)=0$
Hence (1.24) gives

$$(\bar{\lambda}-\lambda)\int_a^b p(x)|y(x)|^2dx=0$$

Since $p(x)\ge 0\; and\; b\ge a$, the integral in the L.H.S is positive.
Hence $(\bar{\lambda}-\lambda)=0$
i.e, $\bar{\lambda}=\lambda$ (or) λ *is real*

1.8. Theorem
The eigenfunctions of a strum-Liouville system corresponding to two different eigenvalues are orthogonal.

Proof:
Let the strum-Liouville system by the second order differential equation

$$\frac{d}{dx}(r\,y')+(q+\lambda p)y=0 \qquad ...(1.25)$$

In the interval $a\le x\le b$ where p, q, r are real functions together with the boundary conditions.

$$a_1y(a)+a_2y'(a)=0 \qquad ...(1.26)$$

$$b_1y(b)+b_2y'(b)=0 \qquad ...(1.27)$$

then a_1, a_2, b_1 and b_2 are real.

Let $y_1(x)$ and $y_2(x)$ be two eigenfunctions corresponding to two different eigenvalues λ_1 and λ_2 of λ

Then

$$\frac{d}{dx}(r\,y_1')+(q+\lambda_1p)y_1=0 \qquad ...(1.28)$$

$$\frac{d}{dx}(r\,y_2')+(q+\lambda_2p)y_2=0 \qquad ...(1.29)$$

Also

$$a_1y_1(a)+a_2y_1'(a)=0 \qquad ...(1.30)$$

$$b_1y_1(b)+b_2y_1'(b)=0 \qquad ...(1.31)$$

$$a_1 y_2(a) + a_2 y_2^{'}(a) = 0 \qquad ...(1.32)$$

$$b_1 y_2(b) + b_2 y_2^{'}(b) = 0 \qquad ...(1.33)$$

(1.29) x y_1 - (1.28) x y_2 gives

$$y_1 \frac{d}{dx}(r\, y_2^{'}) - y_2 \frac{d}{dx}(r y_1^{'}) + p y_1 y_2 (\lambda_2 - \lambda_1) = 0$$

i.e., $$\frac{d}{dx}(r\, y_1 y_2^{'} - r y_2 y_1^{'}) = p y_1 y_2 (\lambda_1 - \lambda_2) \qquad ...(1.34)$$

Integrating both sides of (1.34) with respect to x from a to b

We have,

$$\int_a^b p y_1 y_2 (\lambda_1 - \lambda_2) dx = \left(r y_1 y_2^{'} - r y_2 y_1^{'} \right)_a^b$$

$$= r(b)\left[y_1(b) y_2^{'}(b) - y_2(b) y_1^{'}(b) \right]$$

$$- r(a)\left[y_1(a) y_2^{'}(a) - y_2(a) y_1^{'}(a) \right] \quad ...(1.35)$$

Now (1.30) x $y_2^{'}(a)$ - (1.32) x $y_1^{'}(a)$ gives

$$a_1\left[y_1(a) y_2^{'}(a) - y_2(a) y_1^{'}(a) \right] = 0$$

As $a_1 \neq 0,\ y_1(a) y_2^{'}(a) - y_2(a) y_1^{'}(a) = 0$

Similarly $y_1(b) y_2^{'}(b) - y_2(b) y_1^{'}(b) = 0$

Hence (1.35) gives

$$(\lambda_1 - \lambda_2) \int_a^b p(x) y_1(x) y_2(x) dx = 0$$

But $\lambda_1 \neq \lambda_2$

Hence $$\int_a^b p(x) y_1(x) y_2(x) dx = 0 \qquad ...(1.36)$$

Then (1.36) shows that $y_1(x)$ and $y_2(x)$ are orthogonal in (a, b) with respect to the load function p(x).

Conclusion

In this article we've investigated Strum-Liouvillie problems in eigenvalues and eigenfunctions, eigenvalue and special function are discussed. Eigenfunctions expantion and orthogonality of eigen function are lustrated.

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