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A Classification of Groups of Small Order upto Isomorphism

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ABSTRACT

Here we classified groups of order less than or equal to 15. We proved that there is only one group of order prime up to isomorphism, and that all groups of order prime (P) are abelian groups. This covers groups of order 2,3,5,7,11,13….Again we were able to prove that there are up to isomorphism only two groups of order 2p, where p is prime and p≥3, and this is $Z_{2p} \cong Z_2$ x Z_p. (Where Z represents cyclic group), and D_p (the dihedral group of the p-gon). This covers groups of order 6, 10, 14….. And we proved that up to isomorphism there are only two groups of order P^2. And these are Z_{p^2} and $Z_p \ x \ Z_p$. This covers groups of order 4, 9…..Groups of order P^3 was also dealt with, and we proved that there are up to isomorphism five groups of order P^3. Which are Z_{p^3}, $Z_{p^2} \ x \ Z_p$, $Z_p \ x \ Z_p \ x \ Z_p$, D_{p^3} and Q_{p^3}. This covers for groups of order 8… Sylow's theorem was used to classify groups of order pq, where p and q are two distinct primes. And there is only one group of such order up to isomorphism, which is $Z_{pq} \cong Z_p \ x \ Z_q$. This covers groups of order 15… Sylow's theorem was also used to classify groups of order p^2q and there are only two Abelian groups of such order which are Z_{p^2q} and $Z_p \ x \ Z_p \ x \ Z_q$. This covers order 12. Finally groups of order one are the trivial groups. And all groups of order 1 are abelian because the trivial subgroup of any group is a normal subgroup of that group.

KEYWORDS: *Abelian, cyclic, isomorphism, order, prime.*
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INTRODUCTION

The knowledge of Lagrange theorem and Sylow's theorem are important tools in the classification of groups. The sylow's first theorem helps us to present a group order in the form of $p^m q$ where q doesn't divide p. Lagrange theorem helps us to know the possible divisors of a group. Again the knowledge of the centre of a group, normal subgroups contained in a group as well as direct product of groups helps us to classify whether a group is abelian or non-abelian. Using the notation of Gorenstein [5], any finite Abelian group G is Isomorphic to a direct product of cyclic groups of prime–power order. Berkovich [1], added that this decomposition for G will have the same number of non-trivial factors of each other. For example; $Z_6 \cong Z_2 \times Z_3$, $Z_{12} \cong Z_3 \times Z_4$. The knowledge of normal subgroup is an indispensable tool in the study of group classification. It will aid to differentiate abelian and non-abelian groups. A normal subgroup is a subgroup that is invariant under conjugation by members of the group of which it is a part. In other words, a subgroup N of the group G is normal in G if and only if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$. Written as $N \lhd G$. Evariste Galois was the first to realize the significance of normal subgroups. Dummit [11], normal subgroups are imperative because they (and only they) can be used to create quotient groups of the given group. Fraleigh[12], the normal subgroup of G are specifically the kernels of group homomorphisms with domain G, which implies that they can be used to internally classify those homomorphisms.

The study of the Centre of a group will equally help us to know which groups are Abelian and those that are non – Abelian.

The Centre of a group

Let G be any group. The Centre of G is denoted by Z (G) = {x ∈ G | xg = gx ∀ g ∈ G}

Thus the Centre of G consists of all those elements of G which commute with every element of G. Note: if all element of a group commutes with each other i.e. {Z(G)=G} the Centre of the group is the group itself, we say that the group is Abelian. Nevertheless, there are some groups that its Centre is not the group itself. Those groups are called non-Abelian groups. Roman, S. (2019 unpublished dissertation), noted that all finite Abelian groups are built from cyclic groups of prime-power order using direct product. For symmetric group {Sn} of n ≥ 3 is not an Abelian group. It is also important to note that all cyclic groups denoted by Z are Abelian. This is from our knowledge of center of a group. The Centre of cyclic groups gives us the group itself, which implies that all cyclic groups are Abelian.

1. *Groups of Order Prime*(p) *and* $2p$

***Proposition* 1.1**: *Up to isomorphism there is only one group of order prime.*

***Lemma* 1.2**: *Every Group of Order Prime is a cyclic group, hence has only one generator i.e. itself.*

Proof:
Let g ∈ G be arbitrary chosen
by Lagrange theorem
Let G be a finite group and H a subgroup of G; H < G
Then /H/ divides /G/

|g| = |<g>| will divide |G|=P
hence $|g| = g^1 = e=1$
i.e. g is identity element

or

|g| = p
$g^1 \times g^2 \times \ldots \times g^{p-1}$
$\langle g\rangle = \{g.g^{1 \ldots} g^{p-1}. e\}= G$

⇒ G is cyclic
By G being cyclic, it means it has only one element generator i.e. it has only one element that generates all the other elements of G. Therefore, it's isomorphic to Additive group of integers Modp. G≅Zp

From the proof, we can see that there is only one divisor of G if G is prime. And that is G itself since 1 is the identity. Hence G ≅Zp Iff G is cyclic.

The consequence of this is that groups of order 1, 2, 3, 5, 7, 11, 13… have only one group Up to Isomorphism.

Corollary 1.3: There is only one group of order Prime (P) up to isomorphism and it's Abelian.

***Proposition* 1.4**: *suppose G is a group of order 2p, where* $p \geq 3$, *is a prime.*

Either:
A. $\cong Z_{2p}$ *is a cyclic group or*
B. $G \cong$ D_p *is Isomorphic to the dihedral group of the P − gon.*

Proof:
By disjunctive syllogism i.e. either $G \cong Z_{2p}$ or $G \cong D_p$
suppose G is not cyclic
According to Lagrange theorem
Let G be a finite group and H < G be a subgroup of G. */H/ divides /G/*.
But the only divisors of $2p$ are $1, 2, P, 2P$. But since we assume G is not cyclic then no order 2p can exist. Because if it does exist then it can generate all the other elements of the group so we are left with; $1, 2, P.$ $(1 = identity\ elt\ (e))$

Suppose there is no order of P in the Group i.e. by contradiction then:

All elements of the group would be order 1 or 2

Then we would have an Evolution i.e every non-identity element will be its own inverse ⇒ that G = Abelian.

Let the two elements of G be a & b

G = {e, a, b, ab}

Since the elements of G is closed and contains an identity element and is finite; Hence
{e, a, b, ab} < G i.e. {e, a, b, ab} is a subgroup of G

But it's a subgroup that has four elements.
But G cannot have four elements because 4 does not divide 2p by Lagrange theorem; surely we don't know what 2p is but we know that 2 goes into P once because 2 does not divide P if 4 does not divide 2P, hence /a/ = P; that means the first contradiction cannot hold.

If /a / = p then <a> (generator)
$\Rightarrow a, a^2, a^3 \ldots a^{p-1}$

Let's take another arbitrary element b
If b is not an element of <a>, we can claim that / b/ = 2
If not / b / = P and we don't want it; because by product theorem
$$< a > n < b > = \{e\}/HK/ = /H k/ \times /H n k/ = P^2 \geq 2p$$

i.e. P^2 1 is not contained in 2P, iif H and K are both primes. Hence a contradiction,
So |b| ≠ P
Thus $/b/ = 2$

Finally $ab = (ab)^{-1} = b^{-1} a^{-1} = ba^{-1}$ for all of b an evolution $ab = ba^{-1}$ and from the dihedral group we have that; $D_{2n} = < a, b / a^n = y^2 = 1;\ ab = ba^{-1} >$

The same way a, b relate in the cyclic group of order 2P, is the same way a, b relate in the dihedral group of the p-gon
⇒ G ⇒ dihedral and G ⇒ cyclic
the consequences of this proof is that a group of order 2p is either isomorphic to Z_{2p} or Isomorphic to D_p (i.e. the dihedral group of the p-gon)

The result of this proof implies that there are only two group of order 6, 10, 14…

2. *Groups of Prime Square* (p^2) *and prime cube* (p^3)

***Theorem* 2.1**
There are only two groups of order P^2

***lemma* 2.2**
Let P be prime: there are only two groups Upto Isomorphism of order P^2

Proof:
suppose G is a P^2 group. It is Abelian. According to Lagrange Theorem the divisors of P^2 are 1, P, P^2
Let x: x has order P^2 then G= <x> generates all the elements of the group G = P^2.

So G is cyclic. This satisfies the earlier notation that G is Abelian.
$G \cong Z_{p2}$
Now assume that there is no element of order P^2.

This means that every element which is not the identity has order P. pick x order P. since <x> ≤G, you can take another order P element y in the complement of <x>.

Now
$\Theta: (u, v) \rightarrow uv$
yields a homomorphism from <x> × <y> to G.

Note that <x> n <y> = <e>, so the latter is injective. Since by Lagrange theorem both groups have same cardinality, it follows that Θ is an Isomorphism. If <y> is a complement of <x> it suffices that only the identity element will be the intersection since they are different primes. And of course we all know that the cardinality of primes is always the same. It implies that Θ is an Isomorphism.

Finally since $<x> \cong <y> \cong Z_p$

$G\cong <x> x <y> \cong Z_p x Z_p$

So G is either Isomorphic to Z_p^2 or to $Z_p \times Z_p$ of course the implication of this is that every group of P^2 is either Z_{p2} or $Z_p{\times}Z_p$ i.e. there are only two groups of order P^2 up to Isomorphism.

This covers groups of order 4, 9…

***Proposition* 2. 3**
There are five groups of order P^3 either
1. $G \cong Z_{p3} \cong Z_p \times Z_{p2} \cong Z_p \times Z_p \times Z_p$ Or
2. $G \cong D_{p3} \cong Q_{p\,3}$

Proof:
From the proposition above (2.3) we can deduce that by transitive property that
$G \cong Z_{p3}$
$G \cong Z_P \times Z_{p2}$
$G \cong Z_P \times Z_P \times Z_P$
$G \cong D_P{}^3$
$G \cong Q_P{}^3$

That's five groups in total
by disjunctive syllogism i.e. either / or, Suppose G is not cyclic

by Lagrange theorem
Let G be a finite group and H<G be a subgroup of G but the only divisors of P^3 are
1, P, P^2 , P^3

But we can't take order P^3 because <x> = p^3 will generate all the members of the group making it cyclic. So we have 1, P, P^2

Suppose we take |b| = p; <b> will generate all the members of P and suppose we take |a|= P^2; <a> will generate all the members of P^2. Hence group of order P^3 must contain some cyclic groups.

But let order P^3 have x, y & z; recall G = $Z_p{}^3$ hence |<x>| ≤ G and |<y>| ≤ G also |<z>| ≤ G
$F: (x, y, z) \longrightarrow x \times y \times z$

Let F be a homomorphism that map <x> × <y> × < z > to G
Of course since {x, y, z} ∈ P^2 and also {x, y, z} is contained in P^2

Then <x> n <y> n <z> = e and they must have the same cardinality Iff {x, y, z} are subgroups of order P^2 and are contained in G i.e. G = P^3

then $G \cong <x> \cong <y> \cong <z> \cong Z_P{}^3$
$G\cong <x> \times <y> x < z> \cong Z_P{}^2 x Z_p$
$G \cong <x> \times <y> \times <z> \cong Z_P{\times} Z_P \times Z_P$

But if |b| = p & | a| = P^2; then b = |2|

Hence b is an evolution; therefore being its own inverse $ab = (ab)^{-1} = b^{-1}a^{-1}$

But b is an evolution
⇒ ab = ba^{-1} and from the dihedral group we know that

$D_{2n} = <a, b \,/\, a^n = y^2 = 1; ab = ba^{-1}>$
And also of the Quaternion group

$Q_{4n} = < a, b \,/\, a^{2n} = y^4 = 1; ab = ba^{-1}>$
the same way a, b relate to the cyclic group of order p^3, is the same way a, b relate to the dihedral and Quaternion groups of same order.
G ⇒ dihedral G ⇒ Quartenion & G ⇒ cyclic
This covers groups of order 8, 27…

3. *Groups of order pq and p^2q*

***Proposition* 3. 1**
If G is a group of order pq for some primes, pq such that p>q and q doesn't divide (p – 1) then
$G \cong Z_{pq} \cong Zp \times Zq$

Proof: we can find a unique sylow p and sylow q subgroups of G.
By the third sylow theorem
Let S_q be sylow q & S_p be sylow p
S_p | q and $S_p = 1 + kp$
Since q is a prime the first condition gives $S_p = 1$ or $S_p= q$

Since p>q the second condition implies then that $S_P = 1$
similarly let Sq be the number of sylow q – subgroups of G

We have
S_q | p and $S_q = 1 + kq$
the first condition gives Sq = 1 or Sq = P. If Sq = P
then the first second condition gives P = 1 + Kq, or P- 1 = Kq
this is however impossible since q doesn't divide (p – 1).
Therefore, we have Sq = 1

Another way to see this is:
$Sp \mid q \;\&\; Sp \equiv 1 \bmod p \implies \{1, kp + 1\} \;\forall\; k \in \mathbb{Z}$
$Sq \mid p \;\&\; Sq \equiv 1 \bmod q \implies \{1, Kq + 1\} \;\forall\; k \in \mathbb{Z}$
Sq | P = {1} or {Sp n Sq} = 1
hence since Sq | P =1, Sq = 1 & Sp = 1

It means we have a unique sylow P subgroup and a unique sylow q subgroup. By the second law of Sylow's theorem. Every element of G of order P belongs to the subgroup P and every element of order q belongs to the subgroup Q. It follows that G contains exactly P-1 elements of order P. exactly q-1 elements of order q and one trivial element of order 1. Since for p, q we have

$pq > (p - 1) + (q - 1) + 1$

There are elements of G of order not equal to 1, p or q. any such element must have order pq.

We can assume an element x of order p and y of order q: y is a complement of x
$|<x>| \leq G$ and $|<y>| \leq G$
F: $(x, y) \longrightarrow x \times y$
Let F be a homomorphism from $<x> \times <y>$ to G,

we have the right to do that since $<x>$ n $<y> = \{e\}$. By Lagrange theorem, the divisors of prime (p) are {1 and p}, hence it follows that |x| and |y| have the same cardinality.
It suffices that F is an Isomorphism
$<x> \cong <y> \cong Z_{pq}$
$G \cong <x> \times <y> \cong Z_p \times Z_q$
this covers groups of order 15 …

Corollary 3.2: Every group of order Z_{pq} is Isomorphic to $Z_p \times Z_q$ and there is only one group of order pq.

***Proposition* 3.3**
For every Abelian group of order p^2q;
(i) $G \cong Z_{P^2} \times Zq$
(ii) $G \cong Z_P \times Z_p \times Zq$

Proof:
Suppose G is a finite group of order p^2q
for all p, q distinct primes: p^2 is not congruent to 1 mod p and q is not congruent to1 mod p
then G is Abelian.

By sylow's theorem
np = 1 + kp and it must divide p^2q.

So, 1+ kp / q and because q is not congruent to 1 mod p $\Longrightarrow$ np = 1. This means we have a unique sylow p (G) for an example of p in the group and is normal and also Isomorphic to Z_{P^2} or $Z_P \times Z_P$

Since q does not divide $p^2 - 1$, therefore
nq = 1 + kq is not congruent to p, p^2 . So we also have a normal sylow q (G). Hence G is Abelian
$G \cong Z_{P^2} \times Zq$
$G \cong Z_P \times Z_P \times Z_q$

Another simpler way to see this is;
$n_{p^2} / q = 1 \bmod p^2 = \{1, p, p^2\}$
$n_q / p^2 = 1 \bmod q = \{1, q\} = 1$

Hence sylow q (G) is characteristically normal in G i.e. we have a unique sylow q (G)

Let order p^2q have x^2, y. $\forall G = p^2q$, and let $x^2 \in p^2$ and $y \in q$
$| < x^2 > | \leq G \ also \ | < y > | \leq G$

Suppose Θ: $x^2, y \to x^2 \times y$ is a homomorphism that maps x^2, y to G.

Since y has order prime (q), and p and q are distinct;
$x^2 \cap y = \{e\}$ and $| x^2| = | y |$, hence $x^2 \cong y \cong x \times x \times y$
$G \cong x^2 \times y$
$G \cong x \times x \times y$
Hence; $G \cong Z_{p^2} \text{ x } Z_q$
$G \cong Z_p \text{ x } Z_p \text{ x } Z_q$

This covers abelian groups of order 12.

***Corollary* 3.4**:
There are only two abelian groups of order p^2q, upto isomorphism.

***Remark* 3.5**: There are (up to Isomorphism) exactly three distinct non-abelian groups of order 12: the dihedral group D_6, the alternating groups A_4, and a group T generated by elements a and b such that | a | = 6, $b^2 = a^3$ and $ba = a^{-1}b$.

Griess [4], the group T of order 12 is an example of a dicyclic group. A presentation of the nth dicyclic group, denoted Dic_n , is again by (x | y) where x = { a, b} and Y = $\{a^{2n}, a^nb^{-2}, b^{-1}aba\}$/that is Dic_n is generated by a and b, where a and b satisfy the relations $a^{2n} = e$, $a^n = b^2$, and $b^{-1}ab = a^{-1}$. The group Dic_n is of order 4_n. So the group T is actually the third dicyclic group, Dic_3. Gorenstein [5], noted that the first dicyclic group is Isomorphic to Z_4; for n greater than or equal to 2, Dic_n is non-Abelian. The second dicyclic group is Isomorphic to the quaternions, Q8 $\cong$ Dic_2. When n is a power of 2, Dic_n is Isomorphic to a "generated quaternion group"

4. CONCLUSION

Berkovich and Janko [2], any finite Abelian group G is Isomorphic to a direct product of cyclic groups of prime – power order. Moreover, this decomposition for G has the same number of non-trivial factors of each other

For example;
$Z_6 \cong Z_2 \times Z_3$
$Z_{12} \cong Z_3 \times Z_4$

The study of the Centre of a group and normal subgroups will equally help us to know which groups are Abelian and those that are non – Abelian.

Note: "there is no known formula giving the number of distinct {i.e. non Isomorphic} groups of order n, for every n. however, we have the equipment to classify all groups of order less than or equal to 15. For prime orders 2, 3, 5, 7, 11 and 13, there is only one group to each of these orders. For orders 6, 10, 14 there are two non-Isomorphic groups of order 4. Z_4 and $Z_2 \times Z_2$. There are five groups of order 8, Z_8, $Z_4 \text{ x } Z_2$, $Z_2 \times Z_2 \text{ x } Z_2$, Q_8 and D_4. There are two groups of order 9 as Z_9 and $Z_3 \times Z_3$.

There are five groups of order 12, Z_{12}, $Z_6 \times Z_2$, A_4, D_6 and T. And there is only one group of order 15, Z_{15}.

All cyclic groups are Abelian. This is from our knowledge of Centre of a group. The Centre of cyclic groups gives us the group itself, which implies that all cyclic groups are Abelian.

All finite Abelian groups are built from cyclic groups of prime-power order using direct product. For symmetric group {Sn} of n ≥ 3 is not an Abelian group.

We finish this paper with a table given the known groups of order up to 15.

ORDER	GROUP	COMMENTS
1	Z_1	The Trivial Group
2	Z_2	
3	$Z_3 \cong A_3$	
4	Z_4 Klein 4 – group V $\cong Z_2 \times Z_2$	The Smallest non-cyclic group
5	Z_5	
6	$Z_6 \cong Z_2 \times Z_3$ $S_3 \cong D_3$	The Smallest non – abelian group
7	Z_7	
8	Z_8 $Z_2 \times Z_4$ $Z_2 \times Z_2 \times Z_2$ D_4 Quaternion Q_8	Non- Abelian Non- Abelian
9	Z_9 $Z_3 \times Z_3$	
10	$Z_{10} \cong Z_2 \times Z_5$ D_5	Non – Abelian
11	Z_{11}	
12	$Z_{12} \cong Z_3 \times Z_4$ $Z_2 \times Z_6 \cong Z_2 \times Z_2 \times Z_3$ $D_6 \cong Z_2 \times D_3$ A_4 $Dic_3 \cong T$	Non – Abelian Non Abelian; smallest group which shows converse of Lagrange theorem doesn't hold Non- Abelian, dicyclic group of order 12
13	Z_{13}	
14	$Z_{14} \cong Z_2 \times Z_7$ D_7	Non – Abelian
15	$Z_{15} \cong Z_3 \times Z_5$	

There are 28 groups of order 15 or less, 20 of which are Abelian.

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