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Rp-72: Formulation of Solutions of a Standard Cubic Congruence of Composite Modulus-a Multiple of Four and Power of Three

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ABSTRACT

In this study, a standard cubic congruence of composite modulus --a multiple of four and power of three- is considered. After a thorough study, formulation of the solutions of the said congruence are established in four different cases. It is found that the standard cubic congruence under consideration in case-I has exactly three solutions for an odd positive integer and in case-II, has six solutions for an even positive integer. In case-III, it has nine solutions and in case-IV, the congruence has exactly eighteen solutions. Formulations are the merit of the study.

KEYWORDS: *Composite modulus, cubic residues, Standard cubic Congruence, Formulation*

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INTRODUCTION

The congruence $x^3 \equiv a \pmod m$ is a standard cubic congruence of prime or composite modulus. Such congruence are said to be solvable if a is a cubic residue of m. In this study, the author considered the congruence: $x^3 \equiv a^3 \ (\mathrm{mod}\ 4.3^n)$ for his study and the formulation of its solutions. It is always solvable. But nothing is said to test of the solvability of the said congruence in the literature. It seems that no earlier mathematicians were serious about the said congruence except two: Koshy and Zuckerman.

Koshy only defined a cubic residue [1] while Zuckerman had just started to find the solutions [2] but stopped abruptly without mentioning any method of solutions. Finding no formulation of the solutions of the congruence, the author started his study and formulation of the solutions of cubic congruence Of composite modulus. Many such congruence are formulated and the papers have been published in different International Journals [4], [5], [6]. Here in this paper, the author considered the cubic congruence of the type: $x^3 \equiv a^3 \ (\mathrm{mod}\ 4.3^n)$.

PROBLEM STATEMENT

Here, the problem under consideration is to find a formulation of the solutions of Standard cubic congruence of composite modulus of the type:
$x^3 \equiv a^3 \ (\mathrm{mod}\ 4.3^n)$ in four different cases:
Case-I: If $a \neq 3l$, an odd positive integer;
Case-II: If $a \neq 3l$, an even positive integer;
Case-III: If $a = 3l$, an odd positive integer;
Case-IV: : If $a = 3l$, an even positive integer.

ANALYSIS AND RESULT

Case-I: Let $a \neq 3l$ and odd positive integer.
Now consider the congruence $x^3 \equiv a^3 \ (\mathrm{mod}\ 4.3^n)$.
Then, for the solutions, consider $x \equiv 4.3^{n-1}k + a \ (\mathrm{mod}\ 4.3^n); k = 0, 1, 2 \dots \dots$
Therefore, $x^3 \equiv (4.3^{n-1}k + a)^3 \ (\mathrm{mod}\ 4.3^n)$
$\equiv (4.3^{n-1}k)^3 + 3.(4.3^{n-1}k)^2.a + 3.4.3^{n-1}k.a^2 + a^3 \ (\mathrm{mod}\ 4.3^n)$
$\equiv 4.3^n k\{a^2 + 4.3^{n-1}ka + 4^2.3^{2n-3}k\} + a^3 \ (\mathrm{mod}\ 4.3^n)$, if $a \neq 3m$ is odd;
$\equiv a^3 \ (\mathrm{mod}\ 4.3^n)$.
Thus, it is a solution of the congruence. But for $k = 3,4,5 \dots$,, it is seen that the solution is the same as for $k = 0,1,2$.

Therefore, the congruence has exactly three incongruent solutions:
$x \equiv 4.3^{n-1}k + a \pmod{4.3^n}; k = 0, 1, 2.$

Case-II: Let us consider that if $a \neq 3l$, an even positive integer.
Then consider: $x \equiv 2.3^{n-1}k + a \pmod{4.3^n}; k = 0, 1, 2 \ldots ....$
$x^3 \equiv (2.3^{n-1}k + a)^3 \pmod{4.3^n};$
$\equiv (2.3^{n-1}k)^3 + 3.(2.3^{n-1}k)^2.a + 3.2.3^{n-1}k.a^2 + a^3 \pmod{4.3^n}$
$\equiv 4.3^n k\{b + 3^{n-1}ka + 2^1.3^{2n-3}k\} + a^3 \pmod{4.3^n}$, if a is an even positive integer
$\equiv a^3 \pmod{4.3^n}.$

Thus, $x \equiv 2.3^{n-1}k + a \pmod{4.3^n}$ satisfies the said congruence and it is a solution.
But for
$k = 6$, the solution becomes $x \equiv 2.3^{n-1}.6 + a \pmod{4.3^n}$
$\equiv 4.3^n + a \pmod{4.3^n}$
$\equiv a \pmod{4.3^n}$

Which is the same as for the solution for $k = 0$.
For $k = 7, 8, \ldots \ldots.,$ the solutions are also the same as for $k = 1, 2, ...$ respectively.
Thus, the congruence has exactly six incongruent solutions
$x \equiv 2.3^{n-1}k + a \pmod{4.3^n}; k = 0, 1, 2, 3, 4, 5.$

Case-III: Let us now consider that $a = 3l; l = 1, 3, \ldots \ldots$
Then the congruence is: $x^3 \equiv (3l)^3 \pmod{4.3^n}$
Also consider $x \equiv 4.3^{n-2}k + 3l \pmod{4.3^n}.$
$x^3 \equiv (4.3^{n-2}k + 3l)^3 \pmod{4.3^n}$
$\equiv (4.3^{n-2}k)^3 + 3.(4.3^{n-2}k)^2.(3l) + 3.4.3^{n-2}k.(3l)^2 + (3l)^3 \pmod{4.3^n}$
$\equiv (3l)^3 \pmod{4.3^n}$

Therefore, it is a solution of the congruence. But for $k = 9$, the solution reduces to
$x \equiv 4.3^{n-2}.9 + 3l \pmod{4.3^n}.$
$\equiv 4.3^n + 3l \pmod{4.3^n}.$
$\equiv 3l \pmod{4.3^n}.$

It is a solution of the same congruence as for $k = 0$.

Similarly, it is also seen that $for\ k = 10, 11, \ldots \ldots,$ the solutions are also the same as for $k = 1, 2, \ldots \ldots..$

Therefore, it is concluded that the solutions are:
$x \equiv 4.3^{n-2}k + 3l \pmod{4.3^n};$
$k = 0, 1, 2, \ldots..8.$ These are the nine solutions.

Let us now consider that $a = 3l; l = 2, 4, \ldots \ldots$

Then the congruence is: $x^3 \equiv (3l)^3 \pmod{4.3^n}$
Also consider $x \equiv 2.3^{n-2}k + 3l \pmod{4.3^n}.$
$x^3 \equiv (2.3^{n-2}k + 3l)^3 \pmod{4.3^n}$
$\equiv (2.3^{n-2}k)^3 + 3.(2.3^{n-2}k)^2.(3l) + 3.2.3^{n-2}k.(3l)^2 + (3l)^3 \pmod{4.3^n}$
$\equiv (3l)^3 \pmod{4.3^n}$, if $a = 3l, l = 2, 4, \ldots..$

Therefore, it is a solution of the congruence. But for $k = 18$, the solution reduces to
$x \equiv 2.3^{n-2}.18 + 3l \pmod{4.3^n}.$
$\equiv 4.3^n + 3l \pmod{4.3^n}.$
$\equiv 3l \pmod{4.3^n}.$

It is a solution of the same congruence as for $k = 0$. Similarly, it is also seen that $for\ k = 19, 20, \ldots \ldots,$ the solutions are also the same as for $k = 1, 2, \ldots \ldots..$

Therefore, it is concluded that the solutions are: $x \equiv 2.3^{n-2}k + 3l \pmod{4.3^n};$
$k = 0, 1, 2, \ldots..17.$ These are the eighteen solutions.
Sometimes, in the cubic congruence, the integer a may not be a perfect cube. The readers
have to make it so, by adding multiples of the modulus [1].

ILLUSTRATIONS

Example-1: Consider the congruence: $x^3 \equiv 125 \pmod{4.3^4}.$
It can be written as $x^3 \equiv 125 = 5^3 \pmod{4.3^4}$ with $a = 5$, an odd positive integer.
Such congruence has exactly three solutions which are giving by
$x \equiv 4.3^{n-1}k + a \pmod{4.3^n}; k = 0, 1, 2.$
$\equiv 4.3^3k + 5 \pmod{4.3^4}; k = 0, 1, 2.$
$\equiv 108k + 5 \pmod{324}$
$\equiv 5, 113, 221 \pmod{324}.$

Example-2: Consider the congruence $x^3 \equiv 64 \pmod{324}$.
It can be written as $x^3 \equiv 4^3 \pmod{4.3^4}$ with $a = 4$, an even positive integer.

Then the six solutions are
$x \equiv 2.3^{n-1}k + a \pmod{4.3^n}; k = 0, 1, 2, 3, 4, 5.$
$\equiv 2.3^3k + 4 \pmod{4.3^4}$
$\equiv 54k + 4 \pmod{324}; k = 0, 1, 2, 3, 4, 5.$
$\equiv 4, 58, 112, 166, 220, 274 \pmod{324}.$

Example-3: Consider the congruence $x^3 \equiv 216 \pmod{324}.$

It can be written as $x^3 \equiv 6^3 \pmod{4.3^4}$ with $a = 6 = 3.2$
Then the eighteen solutions are
$x \equiv 2.3^{n-2}k + a \pmod{4.3^n}; k = 0, 1, 2, 3, \ldots.., 17.$
$\equiv 2.3^2k + 6 \pmod{4.3^4}$
$\equiv 18k + 6 \pmod{324}; k = 0, 1, 2, 3, \ldots \ldots \ldots., 17.$
$\equiv 6, 24, 42, 60, 78, 96, 114, 132, 150, 168, 186, 204, 222, 240, 258, 276, 294, 312 \pmod{4.3^4}$

Example-4: Consider the congruence $x^3 \equiv 729 \pmod{972}.$
It can be written as $x^3 \equiv 9^3 \pmod{4.3^5}$ with $a = 9 = 3.3$

Then the nine solutions are
$x \equiv 4.3^{n-2}k + 3l \pmod{4.3^n}; k = 0, 1, 2, 3, \ldots.., 8.$
$\equiv 4.3^3k + 9 \pmod{4.3^5}$
$\equiv 108k + 9 \pmod{972}; k = 0, 1, 2, 3, \ldots \ldots \ldots., 8.$
$\equiv 9, 117, 225, 333, 441, 549, 657, 765, 873 \pmod{4.3^5}$

CONCLUSION

Also, it is concluded that the congruence: $x^3 \equiv a^3 \pmod{4.3^n}$ has exactly three incongruent solutions, if a is an odd positive integer: $x \equiv 4.3^{n-1}k + a \pmod{4.3^n};$
$k = 0, 1, 2.$ But has exactly six incongruent solutions, if a is an even positive integer:
$$x \equiv 2.3^{n-1}k + a \pmod{4.3^n}; k = 0, 1, 2, 3, 4, 5.$$

If a is odd multiple of three, then the congruence has nine solutions
$$x \equiv 4.3^{n-2}k + a \pmod{4.3^n}; k = 0, 1, 2, 3, \ldots \ldots..8.$$

If a is even multiple of three, then the congruence has eighteen solutions

$$x \equiv 2.3^{n-2}k + a \ (\mathrm{mod}\ 4.3^{n});\ k = 0, 1, 2, 3, \dots \dots ..17.$$

MERIT OF THE PAPER

In this paper, the standard cubic congruence of composite modulus-a multiple of four and power of three- is studied for its solutions and is the solutions are formulated.

This lessens the labour of the readers to find the solutions.

This is the merit of the paper.

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