International Journal of Trend in Scientific Research and Development (IJTSRD)
Volume 4 Issue 3, April 2020 Available Online: www.ijtsrd.com e-ISSN: 2456 – 6470

Some New Fixed Point Theorems on S-Metric Spaces

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ABSTRACT
In this paper, we present some new type of contractive mappings and prove new fixed point theorems on S-metric Spaces.

KEYWORDS: *Fixed point, S-Metric spaces, Contractive mapping*

How to cite this paper: J. Gayathri | Dr. K. Sathya "Some New Fixed Point Theorems on S-Metric Spaces" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456-6470, Volume-4 | Issue-3, April 2020, pp.182-186, URL: www.ijtsrd.com/papers/ijtsrd30311.pdf

IJTSRD30311

1. INTRODUCTION

Metric spaces are very important in various area of mathematics such as analysis, topology, applied mathematics etc. so various generalized of metric spaces have been studied and several fixed point results were obtained.

Definition: 1.1
Let X be a nonempty set and S: $X^3 \rightarrow [0, \infty)$ be a function satisfying the following conditions for all x, y, z, a∈ X:

A. s (x, y, z) ≥ 0
B. s (x, y, z) = 0 If and only if x=y=z
C. s (x, y, z) ≤ s (x, x, a) + s (y, y, a) + s (z, z, a)

Then the pair (x, s) is called an s-metric space.

Let (x, s) be an s-metric space and T be a mapping from x into x.

We define,

$$S(T_x, T_x, T_y) < \max \{ s(x, x, y), s(T_x, T_x, x)\ s(T_y, T_y, y), s(T_y, T_y, x)\ s(T_x, T_x, y)\} \qquad s(25)$$

for each x,y ∈ X, x≠y

N. Yilmaz orgur and N. Tas presented the notion of a Cs-mapping and obtained some fixed point theorems using such mappings under (s25) in 1.

Motivated by the above studies, we modify the notion of s-metric spaces and define a new type of contractive mappings.

We introduce new contractive mapping condition (V25), defining the notions of a Cv-mapping on s-metric spaces. Also we give some counter examples and prove some fixed point theorems using the notions of a Cv-mapping.

2. A New type of contractive mappings on s-metric spaces

In this section, we recall some definitions, lemmas, a remark and corollary which are needed in the sequel.

Definition: 2.1
Let (x, s) be an s-metric space and $A \subset x$
A. A subset A of X is called s- bounded if there exist r> 0 such that $s(x, x, y) < r$ for all $x, y \in A$.
B. A sequence $\{x_n\}$ in X converges to x if and only if $s(x_n, x_n, x) \to 0$
as $n \to \infty$ that there exists $n_0 \in N$ such that for all $n \geq n_0$, $s(x_n, x_n, x) < \varepsilon$
for each $\varepsilon > 0$.we denote this by $\lim_{n\to\infty} X_n = X$ or $\lim_{n\to\infty} s(X_n, X_n, X) = 0$.
C. A sequences $\{x_n\}$ in X is called cauchy's sequence if $s(x_n, x_n, x_m) \to 0$
as $n, m \to \infty$. That is there exists $n_o \in N$ such that for all $n, m \geq n_o$.
$s(x_n, x_n, x_m) < \epsilon$ for each $\epsilon > o$.
D. The s-metric space (x, s) is called complete if every Cauchy sequence is convergent.

Lemma: 2.2
Let (x,s) be an s-metric space then $s(x,x,y) = s(y,y,x)$ for all $x,y \in X$ (2.1)

Lemma: 2.3
Let (x, s) be an s-metric spaces if $\{x_n\}$ and $\{y_n\}$ are sequence in X such that $x_n \to x$. $y_n \to y$. Then $s(x_n, x_n, y_n) \to s(x, x, y)$.

Remark:
Every s-metric space is topologically equivalent to a B-metric space.

Definition: 2.4
Let (x, s) be an metric space and T be a mapping of x, we define

$$S(Tx, T_x, T_y) < \max\left\{ s(x, x, y), s(T_x, T_x, x), s(T_y, T_y, y), \frac{s(T_y, T_y, x) + s(T_x, T_x, y)}{2} \right\} \qquad V(25)$$

for any $x, y \in X$, $x \neq y$.

Definition: 2.5
Let (x, s) be an s-metric space T be a mapping from x into x. T is called a cv- mapping on x if for any $x \in X$ and any positive integer $n \geq 2$ satisfying

$$T^i x \neq T^j x, \ 0 \leq i \leq j \leq n-1 \qquad (2.2)$$

We have,

$$S(T^n x, T^n x, T^i x) < \max_{1<i<j\leq n}\left\{ s(T^i x, T^i x, x), \frac{s(T^j x, T^j x, x)}{2} \right\} \qquad (2.3)$$

Theorem: 2.6
Let (x, s) be an S-metric space and T be a mapping from X into X. If T satisfies the condition (V25). Then T is a CV-mapping.

Proof:
Let $x \in X$ and the condition (V25) be satisfied by T.By using mathematical induction (2.2) This condition is true. In fact for n=2 by (V25)

we have,

$$s(T^2x, T^2x, Tx) < \max\left\{ s(Tx, Tx, x), s(T^2x, T^2x, Tx)\ s(Tx, Tx, x), \frac{s(Tx, Tx, Tx) + s(T^2x, T^2x, x)}{2} \right\}$$

And so, $s(T^2x, T^2x, Tx) < \max\left\{ s(Tx, Tx, x), \frac{s(T^2x, T^2x, x)}{2} \right\}$

Hence that the condition (2.3) is proved.

Suppose (2.3) is true for n=k-1, k≥3 and denote
$\alpha = \max_{1\leq i<k-1}\{s(T^i x, T^i x, x)\}$

By the induction hypothesis and the condition (v25)

we find,

$$s(T^k x, T^k x, T^{k-1}x) < \max\left\{ s(T^{k-1}x, T^{k-1}x, T^{k-2}x), s(T^{k-1}x, T^{k-1}x, T^{k-2}x), \frac{s(T^{k-1}x, T^{k-1}x, T^{k-1}x) + s(T^k x, T^k x, T^{k-2}x)}{2} \right\}$$

$$< \max\left\{ \alpha, \frac{s(T^k x, T^k x, T^{k-2}x)}{2} \right\}$$

Also by induction it can be shown that,

$$s(T^k x, T^k x, T^{k-i}x) < \max\{\alpha, s(T^k x, T^k x, T^{ki-1}x)\}$$
$$< \max\{s(T^i x, T^i x, x), s(T^k x, T^k x, T^{ki-1-1}x)\}$$

For i=k-1

$s(T^k x, T^k x, Tx) < \max\{ s(T^{k-1}x, T^{k-1}x, x), \frac{s(T^k x, T^k x, x)}{2} \}$

And hence,

$s(T^k x, T^k x, T^i x) < \max_{1 \le i \le j < k}\{s(T^i x, T^i x, x), \frac{s(T^j x, T^j x, x)}{2}\}$

Hence the condition (2.3) is satisfied. The proof is completed. The converse part of theorem is not true for always. Now we see in the following example.

Examples: 2.7

Let R be the real line. Let us consider the usual s-metric on R is defined in

$s(x, y, z) = |x-z| + |y-z|$ for all $x, y, z \in R$

Let,

$Tx = \{ x$ if $x \in [0,1]$

$x-2$ if $x=4$

1 if $x=2$

Then T is a self mapping on the s-metric space $[0,1] \cup \{2,4\}$.

Solution:

For $x=1/3$, $y=1/4 \in [0,1]$ we have,

$s(Tx, Tx, Ty) = s(1/3, 1/3, 1/4) = 1/6$, $s(x, x, y) = s(1/3, 1/3, 1/4) = 1/6$

$s(Tx, Tx, x) = s(1/3, 1/3, 1/3) = 0$, $s(Ty, Ty, y) = s(1/4, 1/4, 1/4) = 0$

$s(Ty, Ty, x) = s(1/4, 1/4, 1/3) = 1/6$, $s(Tx, Tx, y) = s(1/3, 1/3, 1/4) = 1/6$

And so,

$s(Tx, Tx, Ty) = 1/6 < \max\{ 1/6, 0, 0, 1/6\} = 1/6$ Therefore T does not satisfy the condition (2.2) now we show that T is a CV-mapping we have the following case for $x \in \{2,4\}$.

Case: 1

For $x=2$, $n=2$

$s(T^2, T^2, T_2) < \max\{ s(T_2, T_2, 2), \frac{s(T_2^2, T_2, 2)}{2} \}$

$s(1,1,1) < \max\{ s(1,1,2), \frac{s(1,1,2)}{2} \}$

$0 < \max\{2, 1\} = 2$

For $n>2$ using similar arquments we have to see that condition (lemma 2.1.2) holds.

Case: 2

For $x=4$ and $n \in \{2,3\}$

$s(T^2_4, T^2_4, T4) < \max s(T_4, T_4, 4), \frac{s(T^2_4, T^2_4, 4)}{2} \}$

$s(1,1,2) < \max\{ s(2,2,4), \frac{s(1,1,4)}{2} \}$

$2 < \max\{4, 3\} = 4$

$s(T^3_4, T^3_4, T^2_4) < \max\{ s(T^2_4, T^2_4, T4), \frac{s(T^3_4, T^3_4, T4)}{2} \}$

$s(1,1,1) < \max\{ s(1,1,2), \frac{s(1,1,2)}{2} \}$

$0 < \max\{2,1\} = 2$

For $n>3$ using similar arguments we have to see the condition (2.2) holds. Hence T is a CV-Mapping.

Theorem: 2.8

Let T be a CV-mapping from on S-metric space (x,s) into itself. Then T has a fixed point in X if and only if there exist integers l and m, $l>m\geq 0$ and $x \in X$ satisfying ,

$$T^l x = T^m x \qquad (2.4)$$

If this condition is satisfied, then $T^m x$ is a fixed point of T.

Proof:

Necessary condition; Let $x \in X$ be a fixed point of T.

that is, $Tx_1 = x_1$

Then (2.4) is true with $l=1$, $m=0$

Sufficient condition: suppose there exists a point x∈X and an integers l and m, l>m≥0 such that ,
$T^l x=T^m x$

Without loss of generality,
Assume that l is the minimal integer satisfying $T^k x=T^m x$, k>m. Putting $y=T^m x$ and n=l-m we have,
$T^n x = T^n T^m x = T^{l-m} T^m x$
$= T^{l-m+m} x$
$=T^l x= T^m x$
= y

And n is the minimal such integer satisfying ,
$T^n y=y$, n≥1

Now, we show that y is fixed point of T. Suppose not, that is y is not a fixed point of T. Then n≥2, and
$T^i y \neq T^i y$ for $0 \leq i < j \leq n-1$

Since T is a CV-mapping we have,
s $(T^i y, T^i y,y)$ = $s(T^i y, T^i y, T^n y)$
= s $(T^n y, T^n y, T^i y)$
< max { $s(T^i y, T^i y, y)$, $s(T^i y, T^i y, y)$ }
$1 \leq i < j \leq n$ 2
< max { $s(T^i y, T^i y,y)$, $s(T^i y, T^i y,y)$ }
$1 \leq i < j \leq n$ 2

Then we obtain,
Max $s(T^i y,T^i y, y)$ < max {$s(T^i y, T^i y, y)$, $s(T^j y, T^j y, y)$ }
$1 \leq i < j \leq n-1$ $1 \leq i < j \leq n-1$ 2

That is a contradiction.
Consequently $T^m x = y$ is a fixed point of T.

3. Some Fixed point theorems on s-metric spaces

In this section, we present some fixed point theorems using the notions of a CS-mapping and LS-mapping compactness and diameter on S-metric space.

Theorem: 3.1
Let T be a CS-mapping on X. Then T has a fixed point in X if and only if there exists integers p and q. p>q≥ 0 and a point x∈X such that

$$T^p x = T^q x \qquad (3.1)$$

If the condition (3.1) is satisfied, then $T^q x$ is a fixed point of T.

Proof:
Let X_0 ∈X be a fixed point of T.

ie. $X_0=T_{X_0}$ For P=1, q=0 the condition (3.1) is satisfied.

Conversely, suppose there exists a point x∈X and two integers p,q, p>q≥0 such that, $T^p x = T^q x$

Without loss of generality, we assume that P is the minimal such integer satisfying $T^k x = T^q x$, k>q. Putting $y = T^q x$ and m= p-q,

We have,
$T^m y = T^m.T^q x$
$= T^{p-q+q} x$
$= T^p x$
$= T^q x = y$

And m is the minimal integer satisfying $T^m y = y$, m≥1 Now, we show that y is fixed point of T. Suppose not, that is y is not fixed point of T. Then m≥2 and $T^i y \neq T^j y$, $o \leq i < j \leq m-1$

Since T is a CV-mapping we have,
d $(y,T^i y)$ = d $(T^m y, T^i y)$
<max{ $d(T^i y,y)$, $d(T^j y,y)$ }
$1 \leq i < j \leq m$ 2

<max { $d(T^i y,y)$, $d(T^j y,y)$ }
$1 \leq i < j \leq m-1$ 2

Then we obtain,
Max $d(y,T^i y)$ <max { $d(T^i y,y)$, $d(T^j y,y)$ }
$1 \leq i \leq m-1$ $i < j \leq m-1$ 2

Which is contradiction.
Therefore $y = T^q x$ is a fixed point of T.

Corollary: 3.1
Let (x,d) be an s-metric space and T be a self mapping of X satisfying the condition (2.2). Then T has a fixed point in X if and only if there exist integers p and q, p>q ≥0 and x∈X Satisfying (3.1) If the condition (3.1) is satisfied . Then $T^q x$ is a fixed point of T.

Theorem: 3.2
Let T be an LS-mapping from an s-metric space (x,s) into itself then T has a fixed point in X if and only if there exists integers p and q.

p>q ≥0 and x∈X satisfying (3.1) the condition (3.1) is satisfied. Then $T^q x$ is a fixed point of T.

Proof:
It is obvious from theorem (2.8) and theorem (3.1).

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