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Solving Differential Equations Including Leguerre Polynomial via Laplace Transform

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ABSTRACT
The Laplace transformation is a mathematical tool used in solving the differential equations. Laplace transformation makes it easier to solve the problem in engineering application and make differential equations simple to solve. In this paper, we will solve differential equations including Leguerre Polynomial via Laplace Transform Method.

KEYWORDS: *Laplace Transform, Differential Equation*
SUBAREA: Laplace transformation
BROAD AREA: Mathematics

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INRODUCTION
The Laplace transformation is applied in different areas of science, engineering and technology [1-5]. The Laplace transformation is applicable in so many fields and effectively solving linear differential equations. Ordinary linear differential equation with constant coefficient and variable coefficient can be easily solved by the Laplace transform method without finding their general solutions [6, 7, 8, 9,]. This paper presents the application of Laplace transform in solving the differential equations including Leguerre Polynomial.

DEFINITION
Let F (t) is a well defined function of t for all t ≥ 0. The Laplace transformation [10, 11] of F (t), denoted by f (p) or L {F (t)}, is defined L {F (t)} $=\int_0^\infty e^{-pt}F(t)dt = f(p)$, provided that the integral exists, i.e. convergent. If the integral is convergent for some value of p, then the Laplace transformation of F (t) exists otherwise not. Where p the parameter which may be real or complex number and L is the Laplace transformation operator.

Laplace Transformation of Elementary Functions [12, 13**]**

1. $L\{1\} = \frac{1}{p}, p > 0$
2. $L\{t^n\} = \frac{n!}{p^{n+1}}, where\ n = 0,1,2,3 \dots\dots\dots$
3. $L\{e^{at}\} = \frac{1}{p-a}, , p > a$
4. $L\{sinat\} = \frac{a}{p^2+a^2}, p > a$
5. $L\{sinhat\} = \frac{a}{p^2-a^2}, p > |a|$
6. $L\{cosat\} = \frac{p}{p^2+a^2}, p > 0$
7. $L\{coshat\} = \frac{p}{p^2-a^2}, p > |a|$

Proof: By the definition of Laplace transformation, we know that L {F (t)} $=\int_0^\infty e^{-pt}F(t)dt$ then

$$L\{e^{at}\} = \int_0^\infty e^{-pt}\, e^{at}dt$$

$$= -\frac{1}{p-a}\left(e^{-\infty} - e^{-0}\right) = \frac{1}{p-a}(0-1)$$

$$= \frac{1}{p-a} = f(p), p > a$$

Laplace Transformation of derivatives
Let F is an exponential order, and that F is a continuous and f is piecewise continuous on any interval [14, 15, 16,], then

$$L\{F'(t)\} = \int_0^\infty e^{-pt}F'(t)dt$$

$$= [0 - F(0)] - \int_0^\infty -pe^{-pt}F(t)dt$$

$$= -F(0) + p\int_0^\infty e^{-pt}F(t)dt$$
$$= pL\{F(t)\} - F(0)$$
$$= pf(p) - F(0)$$
Now, since $L\{F'(t)\} = pL\{F(t)\} - F(0)$

Therefore, $L\{F''(t)\} = pL\{F'(t)\} - F'(0)$
$$L\{F''(t)\} = p\{pL\{F(t)\} - F(0)\} - F'(0)$$
$$L\{F''(t)\} = p^2L\{F(t)\} - F(0) - F'(0)$$
$$L\{F''(t)\} = p^2f(p) - F(0) - F'(0)$$
Similarly $L\{F'''(t)\} = p^3f(p) - p^2F(0) - pF'(0) - F''(0)$

And so on.

FORMULATION

Laguerre Polynomial.

The Laguerre polynomial [1-3] is defined as
$$L_n(u) = \frac{e^u}{n!}\frac{d^n}{du^n}(e^{-u}u^n)$$

We know that by the definition of Laplace Transform
$$L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$$

Therefore,
$$L\{L_n(t)\} = \int_0^\infty e^{-pt}\left\{\frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n)\right\}dt$$
$$= \frac{1}{n!}\int_0^\infty e^{-(p-1)t}\left\{\frac{d^n}{dt^n}(e^{-t}t^n)\right\}dt$$
$$= \frac{1}{n!}\left[(p-1)\int_0^\infty e^{-(p-1)t}\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt\right]$$

Integrating again,
$$= \frac{(p-1)^2}{n!}\int_0^\infty e^{-(p-1)t}\frac{d^{n-2}}{dt^{n-2}}(e^{-t}t^n)dt$$

Integrating again,
$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-(p-1)t}\frac{d^{n-n}}{dt^{n-n}}(e^{-t}t^n)dt$$
$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-(p-1)t}(e^{-t}t^n)dt$$
$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-pt}t^n dt$$

But by the definition of Laplace Transformation
$$L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$$

Hence,
$$\frac{(p-1)^n}{n!}L(t^n) = \frac{(p-1)^n}{n!}.\frac{n!}{p^{n+1}}$$

Hence,
$$L\{L_n(t)\} = \frac{(p-1)^n}{p^{n+1}}$$

Solve the differential equations
$$\boldsymbol{(D^2 - 3D + 2)y = L_2(t)}$$
with initial conditions
$$\boldsymbol{y(0) = 1, y'(0) = 0}$$

Solution:

Given equation can be written as
$$y'' - 3y' + 2y = L_2(t)$$

Taking Laplace Transform on sides
$$L\{y''\} - 3L\{y'\} + 2L\{y\} = L\{L_2(t)\}$$

Because Leguerre polynomial of order 2 is
$$L_2\{t\} = \frac{1}{2}\{2 - 4t + t^{2\}}$$
$$[p^2\bar{y}(p) - p\bar{y}(p) - y'(0)] - 3[p\bar{y}(p) - y(0)] + 2\bar{y}(p) = \frac{(p-1)^2}{p^3}$$

Applying initial conditions, we get
$$[p^2 - 3p + 2]\bar{y}(p) - \frac{(p-1)^2}{p^3} + p - 3$$
$$\bar{y}(p) = \frac{p-1}{p^3(p-2)} + \frac{p-3}{(p-1)(p-2)}$$

Applying inverse Laplace Transform
$$y = L^{-1}\left[\frac{p-1}{p^3(p-2)}\right] + L^{-1}\left[\frac{p-3}{(p-1)(p-2)}\right] \dots (1)$$
$$y = U + V \quad \dots\dots\dots\dots(2)$$
$$U = L^{-1}\left[\frac{p-1}{p^3(p-2)}\right]$$

Solving by partial fraction, we get
$$U = -\frac{1}{8}L^{-1}\left[\frac{1}{p}\right] - \frac{1}{4}L^{-1}\left[\frac{1}{p^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{p^3}\right] + \frac{1}{8}L^{-1}\left[\frac{1}{p-2}\right]$$
$$U = -\frac{1}{8} - \frac{1}{4}t + \frac{1}{4}t^2 + \frac{1}{8}e^{2t}$$

And,
$$V = L^{-1}\left[\frac{p-3}{(p-1)(p-2)}\right]$$

Solving by Heaviside's expansion

Let
$$F(p) = p - 3$$
$$G(p) = p^2 - 3p + 2$$

Therefore, $G'(p) = 2p - 3$

Putting $G(p) = 0$, then $p = 1,2$

Here, $G(p)$ have two distinct roots.

Also the degree of $F(p)$ is less than the degree of$G(p)$.

Therefore by Heaviside's expansions
$$V = L^{-1}\left\{\frac{F(p)}{G(p)}\right\}$$
$$= \frac{F(1)}{G'(1)}e^t + \frac{F(2)}{G'(2)}e^{2t}$$
$$V = 2e^t - e^{2t}$$

From (2),
$$y = U + V$$
$$\boldsymbol{y = -\frac{1}{8} - \frac{1}{4}t + \frac{1}{4}t^2 + \frac{1}{8}e^{2t} + 2e^t - e^{2t}}$$

Solve the differential equations

$$(D^2 - D - 2)y = L_1(t)$$

with initial conditions

$$y(0) = -1 , y'(0) = 2$$

Solution:

Given equation can be written as

$$y'' - y' - y = L_1(t)$$

Taking Laplace Transform on sides

$$L\{y''\} - L\{y'\} - 2L\{y\} = L\{L_1(t)\}$$

Because Leguerre polynomial of order 1 is

$$L_2\{t\} = \{1 - t\}$$

$$[p^2\bar{y}(p) - p\bar{y}(p) - y'(0)] - [p\bar{y}(p) - y(0)] - 2\bar{y}(p) = \frac{p-1}{p^2}$$

Applying initial conditions, we get

$$[p^2 - p - 2]\bar{y}(p) - \frac{p-1}{p^2} - p + 3$$

$$\bar{y}(p) = \frac{p-1}{p^2(p-2)(p+1)} - \frac{p-3}{(p+1)(p-2)}$$

Applying inverse Laplace Transform

$$y = L^{-1}\left[\frac{p-1}{p^2(p-2)(p+1)}\right] - L^{-1}\left[\frac{p-3}{(p+1)(p-2)}\right] \dots .(1)$$

$$y = U + V \quad \dots \dots .(2)$$

$$U = L^{-1}\left[\frac{p-1}{p^2(p-2)(p+1)}\right]$$

Solving by partial fraction, we get

$$U = -\frac{3}{4}L^{-1}\left[\frac{1}{p}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{p^2}\right]$$

$$+\frac{2}{3}L^{-1}\left[\frac{1}{p+1}\right] + \frac{1}{12}L^{-1}\left[\frac{1}{p-2}\right]$$

$$U = -\frac{3}{4} + \frac{1}{2}t + \frac{2}{3}e^{-t} + \frac{1}{12}e^{2t}$$

And,

$$V = L^{-1}\left[\frac{p-3}{(p+1)(p-2)}\right]$$

Solving by Heaviside's expansion

Let

$$F(p) = p - 3$$
$$G(p) = p^2 - p - 2$$

Therefore, $G'(p) = 2p - 1$

Putting $G(p) = 0$, then $p = -1,2$

Here, $G(p)$ have two distinct roots.

Also the degree of $F(p)$ is less than the degree of $G(p)$.

Therefore by Heaviside's expansions

$$V = L^{-1}\left\{\frac{F(p)}{G(p)}\right\}$$

$$= \frac{F(-1)}{G'(-1)}e^{t} + \frac{F(2)}{G'(2)}e^{2t}$$

$$V = -\frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}$$

From (2),

$$y = U + V$$

$$\boldsymbol{y = -\frac{3}{4} + \frac{1}{2}t + \frac{2}{3}e^{-t} + \frac{1}{12}e^{2t} - \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}}$$

CONCLUSION

The solutions of differential equations including Leguerre Polynomial via Laplace Transform Method are obtained successfully. It is revealed that the Laplace transform is a very useful mathematical for obtaining the solutions of differential equations including Leguerre Polynomial.

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