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The Existence of Approximate Solutions for Nonlinear Volterra Type Random Integral Equations

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ABSTRACT

In this paper, we prove the existence of solutions as well as approximations of the solutions for the nonlinear Volterra type random integral equations. We rely our results on a newly constructed hybrid fixed point theorem of B. C. Dhage in partially ordered normed linear space.

KEYWORDS: *Approximate solution, Existence, Nonlinear, Random Integral Equations*

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1. INRODUCTION

Consider the nonlinear Volterra type random integral equation is in the form of

$$x(\,t,\omega\,) = h\big(\,t,x(t,\omega)\big) + \int_0^t K(\,\tau,x(\tau,\omega);\,\omega\,)\,d\tau \qquad (1.1)$$

for all $t \geq 0$, and for every $\omega \in \Omega$, the function $x:\mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a random unknown function, the function $h:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a random perturbed term for $\omega \in \Omega$ and this function $h\big(\,t,x(t,\omega)\big)$ known function. The kernel $K(\,\tau,x(\tau,\omega);\,\omega\,)$ is the function from $\mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ and this random kernel defined for the $0 \leq \tau \leq t \leq \infty$ and $\omega \in \Omega$.

This type of nonlinear Volterra type random integral equation has been studied by C. P. Tsokos and W. J. Padgett [4, 5], this type of equation are given very importance in stochastic formulation of a classical chemical kinematics problems [7], the formulation of such type of the chemical kinematics problem models results in the nonlinear Volterra type random integral equation of the form (1.1) and also this type of nonlinear Volterra random integral equation has a wild range of applications in applied mathematics, stochastic process, physical problems, engineering, many physical phenomena in life science and technology [4, 5].

2. Auxiliary Results

In this section, we present here some notations, definitions and preliminary facts that will be used in the proofs of our main results.

Let $\mathbb{X}$ be denote the partially orderd real normed linear space with order relation "$\leq$" and the norm $\|.\|$. The nonlinear Volterra type random integral equation (1.1) is in the space $C(\mathbb{R}_+,\mathbb{R})$be the space of all real valued continuous function defined on $\mathbb{R}_+$ and the space $C(\mathbb{R}_+,\mathbb{R})$ be the Banach space with the suprimum norm defined as

$$\|x\| = sup_{t\in\mathbb{R}_+}|x(t)|$$

and also this space $C(\mathbb{R}_+,\mathbb{R})$ is a separable Banach space, now we define an order relation "$\leq$" in $C(\mathbb{R}_+,\mathbb{R})$ as "$x \leq y$" iff $\|x(t)\| \leq \|y(t)\| \quad \forall\, t \in \mathbb{R}_+$ respectively then clearly the space $C(\mathbb{R}_+,\mathbb{R})$ is a partially ordered Banach space w.r.t. above norm defined and a relation "$\leq$".

Definition 2.1 The set $\mathbb{X}$ is known as a regular space, if $\{x_n\}$ is a nondecreasing sequence in $\mathbb{X}$ such that $x_n \to x^{\#}$ as $n \to \infty$ then $x_n \leq x^{\#}$ for all $n \in \mathbb{N}$.

Clearly, the partially ordered Banach space $C(\mathbb{R}_+,\mathbb{R})$ is regular space and the condition guaranteeing the regularity of any partially ordered Normed linear space $\mathbb{X}$ these may be found in literatures of Nieto and Lopez [6].

Definition 2.2 A mapping $T{:}\,\mathbb{X} \to \mathbb{X}$ is said to be nondecreasing mapping, if it preserves the order relation "$\leq$". "i.e." $if\, x \leq y \iff Tx \leq Ty \qquad \forall x,y \in \mathbb{X}$

Definition 2.3 An operator $T{:}\ \mathbb{X} \to\ \mathbb{X}$ on a Normed linear space $\mathbb{X}$ is said to be a compact operator, if $T(\mathbb{X})$ is relatively compact subset of $\mathbb{X}$ and operator T is said to be a totally bounded, if for any bounded subset S of $\mathbb{X}$, $T(\mathbb{X})$ is relatively compact subset of $\mathbb{X}$, and the operator T is said to be a completely continuous on $\mathbb{X}$, if T is continuous and totally bounded.

Definition 2.4 [2] A mapping $T: \mathbb{X} \to \mathbb{X}$ is said to be partially continuous at a point $x_0 \in \mathbb{X}$, if $\forall \epsilon > 0$, there exist $\delta > 0$ such that $\|\mathrm{Tx} - \mathrm{Tx}_0\| < \epsilon$ whenever x is comparable to x_0 and $\|x - x_0\| < \delta$. A mapping $T: \mathbb{X} \to \mathbb{X}$ is said to be partially continuous on $\mathbb{X}$, if it is partially continuous on each and every point of it. Then it is clear that if T is partially continuous on $\mathbb{X}$ then it is continuous on every chain $\mathfrak{C}$ contained in $\mathbb{X}$ similarly T is said to be partially bounded if $T(\mathfrak{C})$ is bounded for each chain $\mathfrak{C}$ in $\mathbb{X}$. An operator T is said to be uniformly partially bounded, if all chains $T(\mathfrak{C})$ in $\mathbb{X}$ are bounded by a unique constant.

Definition 2.5 [1, 2] An operator $T: \mathbb{X} \to \mathbb{X}$ is said to be partially compact, if $T(\mathfrak{C})$ is a relatively compact subset of $\mathbb{X}$ for all toatally order chains $\mathfrak{C}$ in $\mathbb{X}$. An operator T is said to be partially totally bounded, for any totally order and bounded subset $\mathfrak{C}$ of $\mathbb{X}$, $T(\mathfrak{C})$ is a relatively compact subset of $\mathbb{X}$.

If T is partially continuous and partially totally bounded then it is said to be partially completely continuous on $\mathbb{X}$.

Definition 2.6 [2] The order relation " $\leq$ " and the metric *"d"* on a nonempty set $\mathbb{X}$ are said to be compatible, if $\{x_n\}$ is a monotonic, (i.e., "monotonic nondecreasing or monotonic nonincreasing") sequence in $\mathbb{X}$ and if a subsequence $\{x_{n_k}\}$ of sequence $\{x_n\}$ converges to $x^{\#}$, this gives whthe ole sequence $\{x_n\}$ converges to $x^{\#}$, similarly given $(E, \leq, \|.\|)$ is a partially ordered Normed linear space are said to be compatible, if " $\leq$ " and the metric "d" defined by $\|.\|$ are compatible.

Clearly, as the set $\mathbb{R}$ with usual order relation " $\leq$ " and the norm defined by using supremum on absolute value function have this property, similarly the space $C(\mathbb{R}_+, \mathbb{R})$ with usual order relation " $\leq$ " and the norm defined by using supremum on absolute value function are compatible.

Definition 2.7 [1] Let $(E, \leq, \|.\|)$ be the partially ordered Normed linear space, a mapping $T: \mathbb{X} \to \mathbb{X}$ is said to be partially nonlinear $\mathfrak{D}$ Lipschitz, if there exists a $\mathfrak{D}$ function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|Tx - Ty\| \leq \psi \|x - y\|$$

for all comparable elements $x, y \in \mathbb{X}$, and if $\psi(r) = K$ for $K > 0$, then such T is said to be a partially Lipschitz with Lipschitz constant K. If $K < 1$ then T is said to be a partially contraction with contraction constant K and T is called nonlinear $\mathfrak{D}$ contraction if it is a nonlinear $\mathfrak{D}$ Lipschitz with $\psi(r) < r$ for $r > 0$.

Definition 2.8 B. D. Karande and S. G. Shete [3] An operator A on a Banach space $\mathbb{X}$ into itself is called Compact, if for any bounded subset S of $\mathbb{X}$, $A(S)$ is a relatively compact subset of $\mathbb{X}$, and if A is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Theorem 2.1 ("Arzela-Ascoli theorem") If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $C(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

We use the following hybrid fixed point theorem of B. C. Dhage [2] for proving the existence and approximate solution to nonlinear Volterra type random integral equation.

Theorem 2.2 B. C. Dhage [2] Let $(E, \leq, \|.\|)$ be a regular partially ordered complete normed linear space such that the operator relation " $\leq$ " and the norm $\|.\|$ in E are compatible, let $A, B: E \to E$ be the two nondecreasing operators such that

1. A is a partially bounded and partially nonlinear $\mathfrak{D}$ contraction
2. B is partially continuous and partially compact
3. There exists and element $x_0 \in E$ such that $x_0 \leq Ax_0 + Bx_0$

then the operator equation $Ax + Bx = x$ has a solution $x^{\#}$ in E and the sequence $\{x_n\}$ of successive approximation defined by $x_{n+1} = Ax_n + Bx_n$ for $n = 0,1,2,3 \ldots$. Converges monotonically to $x^{\#}$.

3. Existence Result

In this section, we prove the main result.

We consider the following hypothesis,

H1) The function $h: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous

H2) The function $K: \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous

H3) There exists a constant $M \geq 0 \in I$ such that
$$|K(\tau, x(\tau, \omega);\ \omega)| \leq M, \qquad \forall \tau \in \mathbb{R}_+$$

H4) $|K(\tau, x; \omega)|$ is nondecreasing $\mathbb{X}$ for all $\tau \in \mathbb{R}_+$

H5) The equation 1.1 has a lower solution $u \in C(\mathbb{R}_+, \mathbb{R})$
$$\mathrm{u}(t, \omega) \leq h(t, x(t, \omega)) + \int_0^t K(\tau, x(\tau, \omega);\ \omega)\, d\tau \ \ \forall\, t \in \mathbb{R}_+$$

H6) There exists a constant $K \geq 0$ such that
$$|h(t, x(t, \omega))| \leq K \qquad \forall\, t \in \mathbb{R}_+.$$

H7) The function $h: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} - \{0\}$ is a continuous function and there exist a bounded function $\alpha: \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$|h(t, x(t, \omega)) - h(t, y(t, \omega))| \leq \alpha(t)|x(t, \omega) - y(t, \omega)| \ \forall\, t \in \mathbb{R}_+$$
with $|\alpha(t)| < 1$.

Theorem 3.1: Assume that the hypothesis H1) to H7) holds, then the nonlinear Volterra type random integral equation (1.1) has a solution $x^{\#}$ defined on $\mathbb{R}_+$ and the sequence $\{x_n\}$, *n = 0, 1, 2, 3, …* of the successive approximations defined by

$$x_{n+1}(t, \omega) = h(t, x_n(t, \omega)) + \int_0^t K(\tau, x_n(\tau, \omega);\ \omega)\, d\tau \quad (3.1)$$

where $x_0 = u$ converges monotonically to $x^{\#}$.

Proof: Let $\mathbb{X} = C(\mathbb{R}_+, \mathbb{R})$ be then partially ordered Banach space, we define two operators A and B on $\mathbb{X}$ by

$$Ax = h(t, x(t, \omega))$$
$$Bx = \int_0^t K(\tau, x(\tau, \omega);\ \omega)\, d\tau$$

then the nonlinear Volterra type random integral equation (1.1) is equivalent to the operator equation $Ax + Bx = x$.

To the problem of finding the existence of an approximate solution to the nonlinear Volterra type, random integral equation (1.1) is to just reduce to finding the solution to the operator equation $Ax + Bx = x$ in $\mathbb{X}$, as we define the operators Ax and Bx are two operators on and by hypothesis H1) and H2) functions *"h"* and "K" are continuous on $\mathbb{X}$ for all $t \in \mathbb{R}_+$ and as we know that the addition of two continuous functions is continuous and hence $Ax + Bx = x$ is continuous.

We shall show that the operators Ax and Bx satisfy all the conditions of the theorem 2.2.

Step I: A and B are nondecreasing operators on $\mathbb{X}$.

Let $x, y \in \mathbb{X}$ be any elements such that $x \le y$

then as

$$Ax = h\big(t, x(t,\omega)\big) \le h\big(t, y(t,\omega)\big) = Ay$$

$\Rightarrow$ $x \le y$ we gate $Ax \le Ay$

this shows that the operator A is a nondecreasing operator on $\mathbb{X}$.

Similarly, let $x, y \in \mathbb{X}$ be any elements such that $x \le y$

then as

$$Bx = \int_0^t K(\tau, x(\tau,\omega);\ \omega)\, d\tau \le \int_0^t K(\tau, y(\tau,\omega);\ \omega)\, d\tau = By$$

$\Rightarrow$ $x \le y$ we gate $Bx \le By$

this shows that the operator B is also nondecreasing operators on $\mathbb{X}$.

thus both operators A and B are nondecreasing operators on $\mathbb{X}$.

Step II: A is partially bounded and partially nonlinear $\mathfrak{D}$ contraction on $\mathbb{X}$.

Let $x \in \mathbb{X}$ be any element than by H6)

$$|Ax| = \big|h\big(t, x(t,\omega)\big)\big| \qquad \forall\, t \in \mathbb{R}_+ \le K$$

then by taking supremum on all over $\forall\, t \in \mathbb{R}_+$, we get

$$\|Ax\| \le K$$

$\Rightarrow$ the operator A is bounded operators on $\mathbb{X}$.

Next to show that operator A is partially nonlinear $\mathfrak{D}$ contraction on $\mathbb{X}$.

Let $x, y \in \mathbb{X}$ be any elements such that $x \le y$

then as

$$|Ax(t) - Ay(t)| = \big|h\big(t, x(t,\omega)\big) - h\big(t, y(t,\omega)\big)\big| \le \alpha(t)|x(t,\omega) - y(t,\omega)|$$

by taking supremum on all over $\forall\, t \in \mathbb{R}_+$, we get

$$\|Ax(t) - Ay(t)\| \le |\alpha(t)|\|x(t,\omega) - y(t,\omega)\| \le |\alpha(t)|\|x - y\|$$

provided that $|\alpha(t)| < 1$

hence A is partially nonlinear $\mathfrak{D}$ contraction on $\mathbb{X}$.

Step III: B is partially continuous on $\mathbb{X}$.

Let $\{x_n\}$ be a sequence in a chain $\mathfrak{C}$ such that $\{x_n\}$ converges to x, for every $n \in N$, then by Dominated convergence theorem,

we have,

$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty} \int_0^t K(\tau, x_n(\tau,\omega);\ \omega)\, d\tau = \int_0^t \lim_{n\to\infty} K(\tau, x(\tau,\omega);\ \omega)\, d\tau = \int_0^t K(\tau, x(\tau,\omega);\ \omega)\, d\tau = Bx(t)$$

$\forall\, t \in \mathbb{R}_+$.

this shows that, $Bx_n(t)$ converges monotonically to $Bx(t)$ pointwise on $\mathbb{R}_+$.

Now next, we will show that $\{Bx_n\}$ is an equicontinuous sequence of functions on $\mathbb{X}$.

Let $t_1, t_2 \in \mathbb{R}_+$ be any elements with $t_1 \le t_2$ then,

Consider

$$|Bx_n(t_2) - Bx_n(t_1)| = \left| \int_0^{t_2} K(\tau, x_n(\tau,\omega);\ \omega)\, d\tau - \int_0^{t_1} K(\tau, x_n(\tau,\omega);\ \omega)\, d\tau \right| = \left| \int_{t_1}^{t_2} K(\tau, x_n(\tau,\omega);\ \omega)\, d\tau \right| \le \int_{t_1}^{t_2} |K(\tau, x_n(\tau,\omega);\ \omega)|\, d\tau \le |K(\tau, x_n(\tau,\omega);\ \omega)| \int_{t_1}^{t_2} d\tau \le M(t_2 - t_1)$$

$$\Rightarrow |Bx_n(t_2) - Bx_n(t_1)| \le M(t_2 - t_1)$$

as $t_2 - t_1 \to 0$, then $|Bx_n(t_2) - Bx_n(t_1)| \to 0$ uniformly for every $n \in N$.

this shows that $\{Bx_n\}$ converges uniformly to Bx and hence B is partially continuous on $\mathbb{X}$.

Step IV: B is partially compactor on $\mathbb{X}$.

Let $\mathfrak{C}$ be the arbitrary chain in $\mathbb{X}$, then we will show that $\mathfrak{B}(\mathfrak{C})$ is a uniformly bounded and equicontinuous on $\mathbb{X}$.

Firstly we show that $\mathfrak{B}(\mathfrak{C})$ is uniformly bounded, let $y \in \mathfrak{B}(\mathfrak{C})$ be any element $\Rightarrow$ $\exists\, x \in \mathfrak{C}$ be any element such that $y = Bx$, then $\forall\, t \in \mathbb{R}_+$

Consider,

$$|Bx(t)| = \left| \int_0^t K(\tau, x(\tau,\omega);\ \omega)\, d\tau \right| \le \int_0^t |K(\tau, x(\tau,\omega);\ \omega)| d\tau \le |K(\tau, x(\tau,\omega);\ \omega)| \int_0^t d\tau \le M(t) = r$$

$$\therefore\ |Bx(t)| \le r \qquad \forall\, t \in \mathbb{R}_+,$$

*t*aking suprimum on all over t, we obtain

$$\|Bx(t)\| \le r \qquad \forall\, x \in \mathfrak{C}$$

Hence $\mathfrak{B}(\mathfrak{C})$ is uniformly bounded on subset of $\mathbb{X}$.

Next we show that $\mathfrak{B}(\mathfrak{C})$ is an equicontinuous in $\mathbb{X}$.

Let $t_1, t_2 \in \mathbb{R}_+$ be any element such that $t_1 \le t_2$ then,

consider

$|Bx(t_2) - Bx(t_1)|$

$$= \left| \int_0^{t_2} K(\tau, x(\tau,\omega);\ \omega)\, d\tau - \int_0^{t_1} K(\tau, x(\tau,\omega);\ \omega)\, d\tau \right|$$

$$= \left| \int_{t_1}^{t_2} K(\tau, x(\tau,\omega);\ \omega)\, d\tau \right|$$

$$\leq \int_{t_1}^{t_2} |K(\tau, x(\tau,\omega);\ \omega)|\, d\tau$$

$$\leq |K(\tau, x(\tau,\omega);\ \omega)| \int_{t_1}^{t_2} d\tau$$

$$\leq M(t_2 - t_1)$$

as $t_2 - t_1 \to 0$ we get $|Bx(t_2) - Bx(t_1)| \to 0$ uniformly for every $x \in \mathfrak{C}$.

this shows that $\mathfrak{B}(\mathfrak{C})$ is an equicontinuous in $\mathbb{X}$.

$\therefore$ $\mathfrak{B}(\mathfrak{C})$ is a uniformly bounded and equicontinuous set of functions in $\mathbb{X}$ and so it is a compact operator and consequently B is a partially compactor operator on $\mathbb{X}$.

Step V: The operator "u" satisfies the operator inequality "$u \leq Ax + Bx$"

By hypothesis, H5), the nonlinear Volterra type random integral equation 1.1 has a lower solution "u" defined on $\mathbb{R}_+$ as

$$u(t,\omega) \leq h(t, x(t,\omega)) + \int_0^t K(\tau, x(\tau,\omega);\ \omega)\, d\tau$$

$\forall\, t \in \mathbb{R}_+$, thus operator A and B satisfy all the conditions of theorem 3.1 and we apply these results to conclude that the operator equation $Ax + Bx = x$ has a solution defined on $\mathbb{R}_+$.

Consequently, the nonlinear Volterra type random integral equation 1.1 has a solution $x^{\#}$ defined on $\mathbb{R}_+$, furthermore the sequence $\{x_n\}$ of successive approximation defined by (3.1) converges monotonically to $x^{\#}$.

This completes the Proof.

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