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Application of Vertex Colorings with Some Interesting Graphs

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ABSTRACT

Firstly, basic concepts of graph and vertex colorings are introduced. Then, some interesting graphs with vertex colorings are presented. A vertex coloring of graph G is an assignment of colors to the vertices of G. And then by using proper vertex coloring, some interesting graphs are described. By using some applications of vertex colorings, two problems is presented interestingly. The vertex coloring is the starting point of graph coloring. The chromatic number for some interesting graphs and some results are studied.

KEYWORDS: *graph, vertices, edges, the chromatic number, vertex coloring*

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I. INTRODUCTION:

Each map in the plane can be represented by a graph. To set up this correspondence, each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one point are not considered adjacent. The resulting graph is called the dual graph of the map. By the way in which dual graphs of maps are constructed, it is clear that any map in the plane has a planar dual graph. The problem of coloring the regions of the map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color.

II. BASIC CONCEPTS AND DEFINITIONS

A. Basic Concepts

A **graph** G = (V, E) consists of a finite nonempty set V, called the set of **vertices** and a set E of unordered pairs of distinct vertices, called the set of edges. Two vertices of a graph are **adjacent** if they are joined by an edge. The **degree** of vertex v in a graph denoted by d(v) is the number of edges incident to v. We denoted by $\delta(G)$ and $\nabla(G)$ the **minimum and maximum** degree, respectively of vertices of G. If e has just one endpoint, e is called a loop. If e_1 and e_2 are two different edges that have the same endpoints, then we call e_1 and e_2 parallel edges. A graph is simple if it has no loops and **parallel edges.**

The graph $G' = (V', E')$ is a **sub graph** of the graph G = (V, E) if $V' \subseteq V$ and $E' \subseteq E$. In this case we write $G' \subseteq G$. $G' \subseteq G$ but $G' \neq G$ we write $G' \subset G$ and call G' is a **proper sub graph** of G. The removal of a vertex v_i from a graph G results in that subgraph of G consisting of all vertices of G expect v_i and all edges not incident with v_i. This subgraph is denoted by $\boldsymbol{G - v_i}$. Thus $G - v_i$ is the maximal subgraph of G not containing v_i. A **walk** in G is a finite non-null sequence W=$v_0 e_1 v_1 e_2 v_2 \dots v_k e_k$, whose terms are alternately vertices and edges, such that, for 1≤ i≤ k, the ends of e_i are $v_{i-1}\, and\, v_i$. We say that W is a walk from$v_0\ to\ v_k$, or $v_0\ v_k$**-walk.** The vertices $v_0\ and\ v_k$ are called the **origin** and **terminus** of W respectively, and $v_1, v_2, \dots, v_{k-1}$ are its **internal vertices.** The integer k, the number of edges in it is the **length** of W. A walk is called a **path** if there are no vertex repetitions. A $v_0\ v_k$-walk is said to be **closed** if $v_0 = v_k$. A closed walk is said to be **k-cycle**; a k- cycle is **odd** or **even** according to k is odd or even.

Two vertices u and v of G are said to be **connected** if there is a uv-path in G. If there is a uv-path in G for any distinct vertices u and v of G, then G is said to be connected**,** otherwise G is disconnected. If vertices u and v are connected in G, the distance between u and v in G, denoted by d_G (u,v), is the length of a shortest uv- path in G.

B. Vertex Colorings

A **vertex coloring** of a graph is an assignment $f: V \to C$ from its vertex set to a codomain set C whose element are called colors.

C. Vertex k- coloring

For any positive integer k, a **vertex k-coloring** is a vertex-coloring that uses exactly k different colors.

D. Proper Vertex Coloring

A **proper vertex coloring** of a graph is a vertex- coloring such that the endpoints of each edge are assigned two different colors. A graph is said to be vertex k-colorable if it has a proper vertex k- coloring.

E. Chromatic Number
The vertex **chromatic number** of G, denoted by, is the minimum number different colors required for a proper vertex- coloring of G.

F. Simple Graph
A graph which has neither loops nor multiple edges. i.e, where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a **simple graph**.

G. Example
A **simple graph** G and its coloring are shown in Figure 1(a) and 1(b).

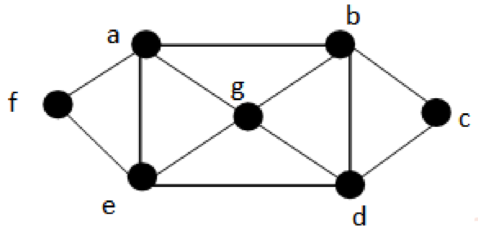

Fig. 1(a) the Simple Graph G

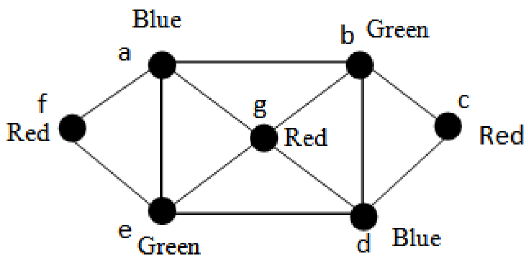

Fig. 1(b) Coloring of the Simple Graph G

H. Example
The graph G by using proper vertex 4- coloring is shown in figure.

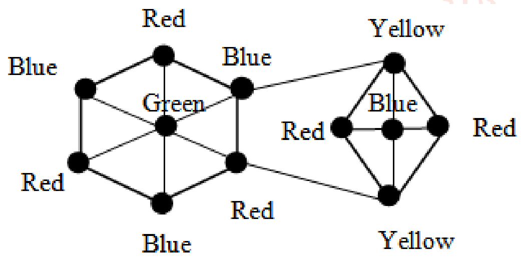

Fig.2 A Proper Vertex 4-coloring of a Graph G

I. Example
We can find the chromatic number of the graph G. Coloring of the graph G with the chromatic number 3 is shown in figure.

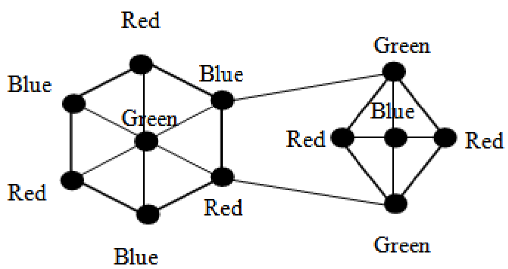

Fig.3 Coloring of the Graph G with the Chromatic Number = 3

III. SOME INTERESTING GRAPH AND COLORING OF GRAPHS
The planar graph and by using proper vertex coloring, some interesting graphs are described in the following.

A. Definitions
Let G = (V, E) be a graph. A **planar embedding** of G is a picture of G in the plane such that the curves or line segments that represent edges intersect only at their endpoints. A graph that has a plane embedding is called a **planar graph.**

B. Example
The planar graph and the planar embedding of graph are shown in figure.

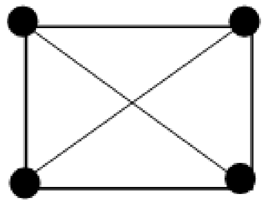

Fig. 4(a) A Planar Graph

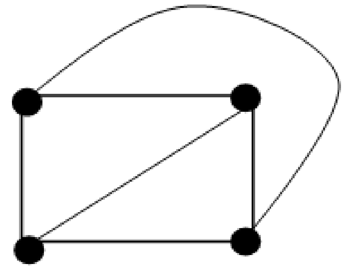

Fig. 4(b) A Planar Embedding of G

C. Definitions
Region of a planar embedding of a graph is a maximal connected portion of the plane that remains when the curves, line segments, and points representing the graph are removed.

Unbounded Region in the planar embedding of a graph is the one region that contains points arbitrary far away from the graph.

A component of G is a maximal connected sub graph of G.

D. Induced Sub graph
For any set S of vertices of G, the **induced sub graph** $\langle S\rangle$ is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in $\langle S\rangle$ if and only if if they are adjacent in G.

E. Theorem
Every planar graph is 6 colorable.

Proof
We prove the theorem by induction on the number of vertices, the result being trivial for planar graphs with fewer than seven vertices. Suppose then that G is a planar graph with n vertices, and that all planar graphs with n-1 vertices are 6-colorable. Without loss of generality G can be assumed to be a simple graph, and so, by Lemma (If G is a planar graph, then G contains a vertex whose degree is at most 5) contains a vertex v whose degree is at most 5; if we delete v, then the graph which remains has n-1 vertices and is that 6-colorable. A 6-coloring for G is then obtained by coloring v with a different color from the (at most 5) vertices adjacent to v.

F. Bipartite Graph
A graph G = (V, E) is **bipartite** if V can be partitioned into two subsets $V_1$and V_2(i.e, V = $V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$),

such that every edges $e \in E$ joins a vertex in V_1 and a vertex in V_2.

G. Example
The bipartite graph is shown in figure.

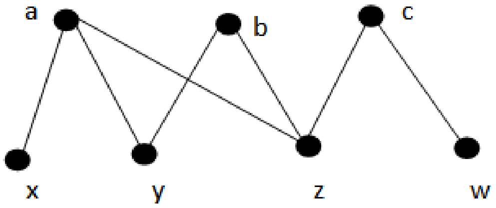

Fig. 5 the Bipartite Graph

H. Theorem
The graph is bipartite if and only if its chromatic number is at most 2.

Proof
If the graph G is bipartite, with parts V_1 and V_2, then a coloring of G can be obtained by assigning red to each vertex in V_1 and blue to each vertex in V_2. Since there is no edge from any vertex in V_1 to any other vertex in V_1, and no edge from any vertex in V_2 to any other vertex in V_2, no two adjacent vertices can receive the same color. Therefore, the chromatic number of bipartite graph is at most 2.

Conversely, if we have a 2-coloring of a graph G, let V_1 be the set of vertices that receive color 1 and let V_2 be the set of vertices that receive color 2. Since adjacent vertices must receive different colors, there is no edge between any two vertices in the same part. Thus, the graph is bipartite.

I. Example
A coloring of bipartite graph with two colors is displayed in Figure.

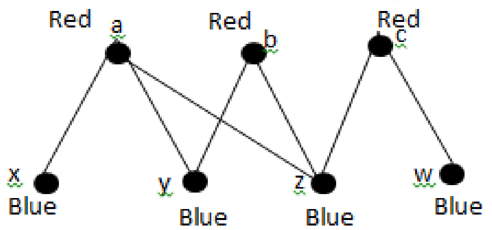

Fig. 6 Coloring of Bipartite Graph

J. Complete Graph
The **complete graph** on n vertices, for n ≥ 1, which we denote K_n, is a graph with n vertices and an edge joining every pair of distinct vertices.

K. Example
The complete graph is shown in figure.

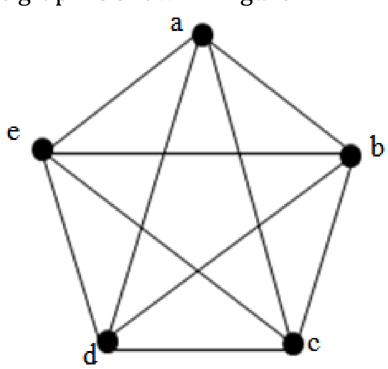

Fig. 7 the Complete Graph

L. Example
We can find the chromatic number of K_n. A coloring of K_n can be constructed using n colors by assigning a different color to each vertex. No two vertices can be assigned the same color, since every two vertices of this graph are adjacent. Hence, the chromatic number of $K_n = n$.

M. Example
A coloring of K_5using five colors is shown in Figure.

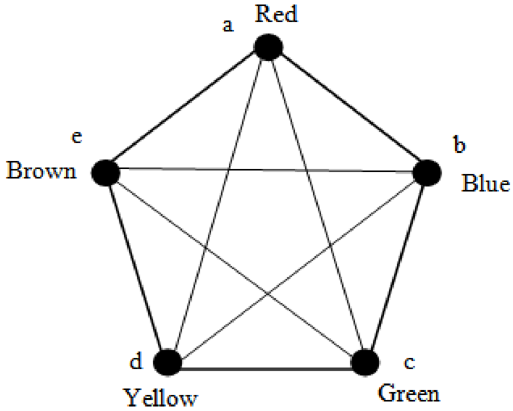

Fig. 8 A Coloring of K_5.

N. Complete Bipartite Graph
The complete bipartite graph $K_{m,n}$, where m and n are positive integers, is the graph whose vertex set is the union V= $V_1 \cup V_2$ of disjoint sets of cardinalities m and n, respectively, and whose edge set is $\{uv|u \in V_1, v \in V_2\}$.

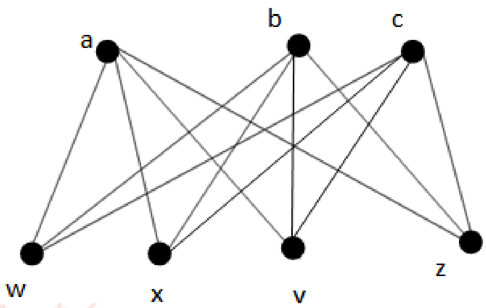

Fig.9The Complete Bipartite Graph

O. Example
A coloring of $K_{3,4}$ with two colors is displayed in Figure.

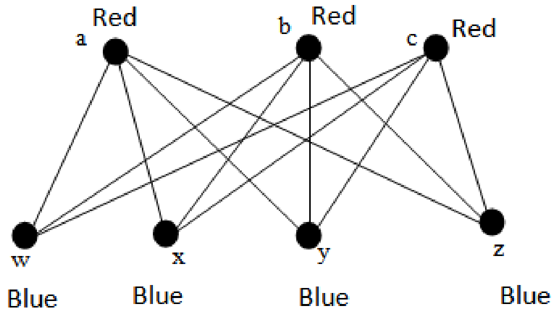

Fig. 10 a Coloring of $K_{3,4}$

P. Example
We can find the chromatic number of complete bipartite graph $K_{m,n}$, where m and n are positive integers. The number of colors needed may seem to depend on m and n. However, only two colors are needed. Color the set of m vertices with one color and the set of n vertices with a second color. Since edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices, no two adjacent vertices have the same color.

Q. Definition
The **n- cycle**, for $n \geq 3$, denoted C_n, is the graph with n vertices, $v_1, v_2, \ldots, v_n$, and edge set E=$\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

R. Example
A coloring of C_6 using two colors is shown in Figure 11.

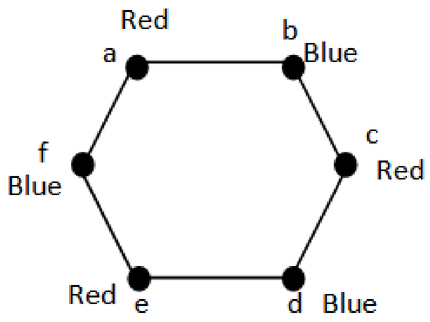

Fig. 11 A Coloring of C_6

A coloring of C_5 using three colors is shown in Figure 12.

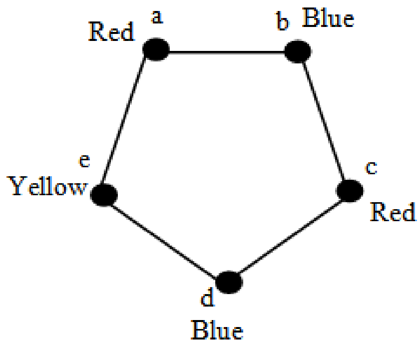

Fig. 12 a coloring of C_5

S. Theorem
The chromatic number of the n- cycle is given by

$$\chi(C_n) = \begin{cases} 2, n = even \\ 3, n = odd \end{cases}.$$

Proof
Let n be even. Pick a vertex and color it red. Proceed around the graph in a clockwise direction (using a planar representation of the graph) coloring the second vertex blue, the third vertex red, and so on. We observe that odd number vertices are colored with red. Since n is even, (n-1) is odd, Thus (n-1)st vertex is colored by red. The nth vertex can be colored blue, since it is adjacent to the first vertex and (n-1)st vertex. Hence, the chromatic number of C_n is 2 when n is even.

Let n be odd. Since n is odd, (n-1) is even. For the vertices of first to (n-1)st, we need only two colors. By first part, we see that the color of first vertex and (n-1)st vertex are not the same. The nth vertex is adjacent to two vertices of different colors, namely, the first and (n-1)st. Hence, we need third color for nth vertex. Thus, the chromatic number of C_n is 3 when n is odd and n>1.

T. Theorem
A graph is 2-colorable if and only if it has no odd cycles.

Proof
If G has an odd cycle, then clearly it is not 2- colorable. Suppose then that G has no odd cycle. Choose any vertex v and partition V into two sets V_1 *and* V_2 as follows,

$$V_1 = \{u \in V : d(u,v) is even\}$$
$$V_2 = \{u \in V : d(u,v) is odd\}$$

Coloring the vertices in V_1 by one color and the vertices in V_2 by another color defines a 2- coloring as long as there are no adjacent vertices within either V_1 or V_2. Since G has no odd cycle, there cannot be adjacent vertices within V_1 or V_2.

U. Definitions
The **n-spoked** wheel, for $n \geq 3$, denoted W_n, is a graph obtained from n-cycle by adding a new vertex and edges joining it to all the vertices of the n- cycle; the new edges are called the **spokes** of the wheel.

V. Example
A coloring of six-spoked wheel using three colors is shown in Figure.

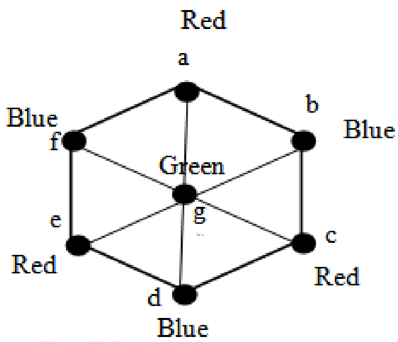

Fig. 13 A Coloring of W_6

W. Example
A coloring of five- spoked wheel using four colors is shown in Figure.

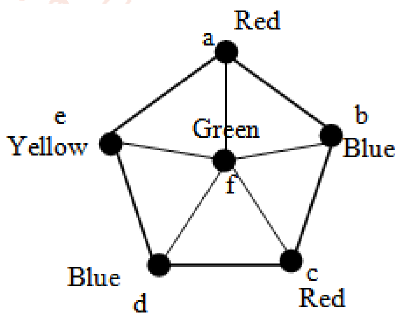

Fig. 14 A Coloring of W_5

X. Proposition
Wheel graphs with an even number of 'spoke' can be 3-colored, while wheels with an odd number of 'spoke' require four colors.

Y. Theorem
If G is a graph whose largest vertex- degree is then G is $(\rho + 1)$-colorable.
Proof

The proof is by induction on the number of vertices of G. Let G be a graph with n vertices; then if we delete any vertex v (and the edges incident to it), the graph which remains is a graph with n-1 vertices whose largest vertex–degree is at most ρ. By our induction hypothesis, this graph is $(\rho + 1)$-colorable; a $(\rho + 1)$- coloring for G is then obtained by coloring v with a different color from the (at most ρ) vertices adjacent to v.

IV. SOME APPLICATIONS OF VERTEX COLORINGS

The followings are some applications of vertex colorings.

A. Example

Suppose that Mg Aung, Mg Ba, Mg Chit, Mg Hla, Mg Kyaw, Mg Lin and Mg Tin are planning a ball. Mg Aung, Mg Chit, Mg Hla constitute the publicity committee. Mg Chit, Mg Hla and Mg Lin are the refreshment committee. Mg Lin and Mg Aung make up the facilities committee. Mg Ba, Mg Chit, Mg Hla, and Mg Kyaw form the decorations committee. Mg Kyaw, Mg Aung, and Mg Tin are the music committee. Mg Kyaw, Mg Lin, Mg Chit form the cleanup committee. We can find the number of meeting times, in order for each committee to meet once.

We will construct a graph model with one vertex for each committee. Each vertex is labeled with the first letter of the name of the committee that is represents. Two vertices are adjacent whenever the committees have at least one member in common. For example, P is adjacent to R, since Mg Chit is on both the publicity committee and the refreshment committee.

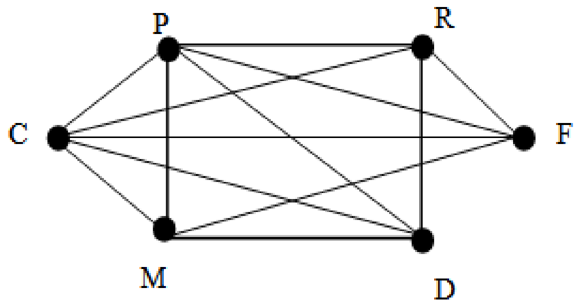

Fig. 15(a) Graph Model for the Ball Committee Problem

To solve this problem, we have to find the chromatic number of G. Each of the four vertices P, D, M, and C is adjacent to each of the other. Thus in any coloring of G these four vertices must receive different colors say 1, 2, 3 and 4, respectively, as shown in Figure. Now R is adjacent to all of these except M, so it cannot receive the colors 1, 2, or 4. Similarly, vertex F cannot receive colors 1, 3, and 4, nor can it receive the color that R receives. But we could color R with color 3 and F with color 2.

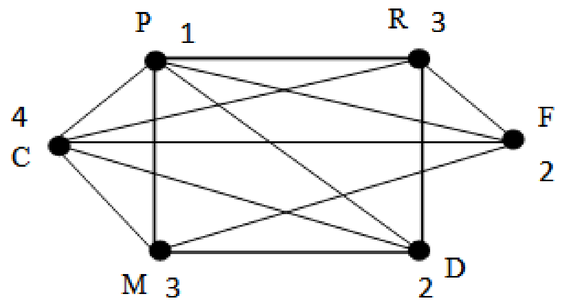

Fig. 15(b) Using a Coloring to Ball Committee Problem

This completes a 4-coloring of G. Since the chromatic number of this graph is 4, four meeting times are required. The publicity committee meets in the first time slot, the facilities committee and the decoration committee meets in the second time slot, refreshment and music committee meet in the third time slot, and the cleanup committee meets last.

B. Example

We can find a schedule of the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math 115 and CS 473, both Math 116 and CS 473, both Math 195 and CS 101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and Math 185 and Math 195, but there are students in every other combination of courses. This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent. Each time slot for a final exam is represented by a different color. A scheduling of the exams corresponds to a coloring of the associated graph.

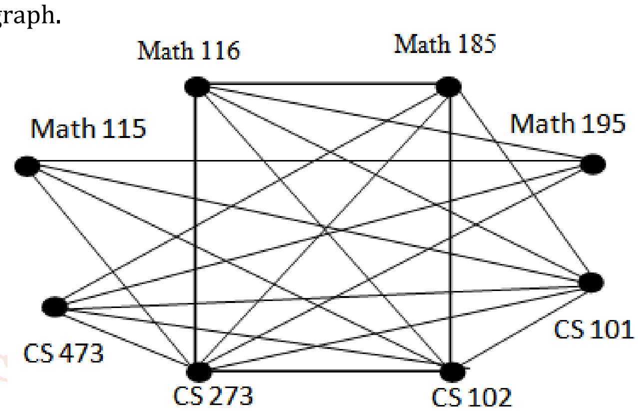

Fig. 16(a) The Graph Representing the Scheduling of Final Exams

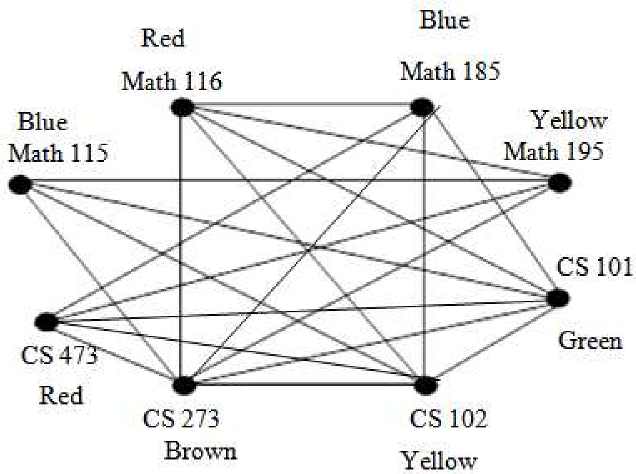

Fig. 16(b) Using a Coloring to Schedule Final Exams

Time Period	Courses
I	Math116, CS 473
II	Math 115, Math 185
III	Math 195, CS 102
IV	CS 101
V	CS 273

Since{Math 116, Math 185, CS 101, CS 102, CS 273}, from a complete sub graph of G, $\chi(G) \geq 5$. Thus five time slots are needed. A coloring of the graph using five colors and the associated schedule are shown in Fig. 16(b).

V. CONCLUSIONS

In conclusion, by using vertex colorings, we can find meeting time slot, Schedules for exams, the Channel Assignment Problem with the corresponding graph models. In Graph Theory, graph coloring is a special case of Graph labeling (Calculable problems that satisfy a numerical condition), assignment of colors to certain objects in a graph subject to certain constraints vertex coloring, edge coloring and face coloring. One of them, vertex coloring is presented. Vertex coloring is used in a wide variety of scientific and engineering problems. Therefore I will continue to study other interesting graphs and graph coloring.

ACKNOWLEDGMENT

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