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The Super Stability of Third Order Linear Ordinary Differential Homogeneous Equation with Boundary Condition

Dawit Kechine Menbiko

Department of Mathematics, College of Natural and Computational Science, Wachemo University, Ethiopia

ABSTRACT

The stability problem is a fundamental issue in the design of any distributed systems like local area networks, multiprocessor systems, distribution computation and multidimensional queuing systems. In Mathematics stability theory addresses the stability solutions of differential, integral and other equations, and trajectories of dynamical systems under small perturbations of initial conditions. Differential equations describe many mathematical models of a great interest in Economics, Control theory, Engineering, Biology, Physics and to many areas of interest.

In this study the recent work of Jinghao Huang, Qusuay. H. Alqifiary, and Yongjin Li in establishing the super stability of differential equation of second order with boundary condition was extended to establish the super stability of differential equation third order with boundary condition.

KEYWORDS: *supper stability, boundary conditions, Intial conditions*

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INTRODUCTION

In recent years, a great deal of work has been done on various aspects of differential equations of third order. Third order differential equations describe many mathematical models of great iterest in engineering, biology and physics. Equation of the form $x''' + a(x)x'' + b(x)x' + c(x)x = f(t)$ arise in the study of entry-flow phenomena, a problem of hydrodynamics which is of considerable importance in many branches of engineering.

There are different problems concerning third order differential equations which have drawn the attention of researchers throughout the world. [17]

In mathematics stability theory addresses the stability of solutions of differential equations, Integral equations, including other equations and trajectories of dynamical systems under small perturbations. Following this, stability means that the trajectories do not change too much under small perturbations [7].The stability problem is a fundamental issue in the design of any distributed systems like local area networks, multiprocessor systems, mega computations and multidimensional queuing systems and others. In the field of economics, stability is achieved by avoiding or limiting fluctuations in production, employment and price.

The stability problem in mathematics started by Poland mathematician Stan Ulam for functional equations around 1940; and the partial solution of Hyers to the Ulam's problem [2] and [20].

In 1940, Ulam [21] posed a problem concerning the stability of functional equations:
"Give conditions in order for a linear function near an approximately linear function to exist."

A year later, Hyers answered to the problem of Ulam for additive functions defined on Banach space: let X_1 and X_2 be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X_1 \to X_2$ Satisfying

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \varepsilon \qquad x, y \in X_1$$

There exists a unique additive function $A : X_1 \to X_2$ with the property

$$\left\| f(x) - A(x) \right\| \le \varepsilon \qquad x \in X_1$$

Thereafter, Rassias [14] attempted to solve the stability problem of the Cauchy additive functional equations in more general setting. A generalization of Ulam's problem is recently proposed by replacing functional equations with differential equations $\varphi(f, y, y', \ldots y^{(n)}) = 0$ and has the Hyers-Ulam stability if for a given $\varepsilon > 0$ and a function y such that

$$\left|\varphi\left(f, y, y', \ldots y^{(n)}\right)\right| \le \varepsilon$$

There exists a solution y_0 of the differential equation such that

$|y(t) - y_0(t)| \le k(\varepsilon)$ And $\lim_{\varepsilon \to 0} k(\varepsilon) = 0$

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher order with constant coefficients in [8, 18] and in [19] respectively.

Rus investigated the Hyers-Ulam stability of differential and integral equations using the Granwall lemma and the technique of weakly Picard operators [15, 16].

Miura et al [13] proved the Hyers-Ulam stability of the first-order linear differential equations

$y'(t) + g(t)y(t) = 0$,

Where $g(t)$ is a continuous function, while Jung (4) proved the Hyers-Ulam stability of differential equations of the form

$\varphi(t)y'(t) = y(t)$

Motivation of this study comes from the work of Li [5] where he established the stability of linear differential equations of second order in the sense of the Hyers and Ulam.

$y' = \lambda y$

Li and Shen [6] proved the stability of non-homogeneous linear differential equation of second order in the sense of the Hyers and Ulam

$y'' + p(x)y' + q(x)y + r(x) = 0$

While Gavaruta et al [1] proved the Hyers- Ulam stability of the equation

$y'' + \beta(x)y(x) = 0$

with boundary and initial conditions.

The recently, introduced notion of super stability [10,11,12,13] is utilized in numerous applications of the automatic control theory such as robust analysis, design of static output feedback, simultaneous stabilization, robust stabilization, and disturbance attenuation.

Recently, Jinghao Huang, Qusuay H. Alqifiary, Yongjin Li[3] established the super stability of differential equations of second order with boundary conditions or with initial conditions as well as the super stability of differential equations of higher order in the form $y^{(n)}(x) + \beta(x)y(x) = 0$ with initial conditions, $y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0$

This study aimed to extend the super stability of second order to the third order linear ordinary differential homogeneous equations with boundary condition.

Main result

Super stability with boundary condition

Definition: Assume that for any function $y \in c^n[a,b]$,if y satisfies the differential inequalities

$$\left|\varphi\left(f, y, y', \ldots, y^{(n)}\right)\right| \le \varepsilon \ldots\ldots(1)$$

for all $x \in [a, b]$ and for some $\varepsilon \ge 0$ with boundary conditions, then either y is a solution of the differential equation

$\varphi(f, y, y', \ldots, y^{(n)}) = 0 \ldots\ldots \quad \ldots\ldots(2)$

or $|y(x)| \le k\varepsilon$ for any $x \in [a,b]$, where k is a constant. Then, we say that (2) has super stability with boundary conditions.

Preliminaries

Lemma 1[10]. Let $y \in c^2[a,b]$, $y(a) = 0 = y(b)$,then

$$\max|y(x)| \le \frac{(b-a)^2}{8} \max|y''(x)|.$$

Proof

Let M = $\max\{|y(x)| : x \in [a,b]\}$ Since $y(a) = 0 = y(b)$, there exists $x_0 \in (a,b)$ such that $|y(x_0)| = M$. By Taylor's formula, we have

$$y(a) = y(x_0) + y'(x_0)(x_0 - a) + \frac{y''(\delta)}{2!}(x_0 - a)^2,$$

$$y(b) = y(x_0) + y'(x_0)(b - x_0) + \frac{y''(\eta)}{2!}(b - x_0)^2,$$

Where $\delta, \eta \in (a,b)$

Thus $|y''(\delta)| = \dfrac{2M}{(x_0 - a)^2}$,

$$|y''(\eta)| = \frac{2M}{(b - x_0)^2}$$

In the case $x_0 \in \left(a, \dfrac{(a+b)}{2}\right]$, we have

$$\frac{2M}{(x_0-a)^2} \ge \frac{2M}{\frac{(b-a)^2}{4}} = \frac{8M}{(b-a)^2}$$

In the case $x_0 \in \left[\dfrac{a+b}{2}, b\right)$, we have

$$\frac{2M}{(b-x_0)^2} \ge \frac{2M}{\frac{(b-a)^2}{4}} = \frac{8M}{(b-a)^2}$$

So, $\max|y''(x)| \ge \dfrac{8M}{(b-a)^2} = \dfrac{8}{(b-a)^2}\max|y(x)|$

Therefore, $\max|y(x)| \le \dfrac{(b-a)^2}{8}\max|y''(x)|$

In (2011) the three researchers Pasc Gavaruta, Soon-Mo, Jung and Yongjin Li, investigate the Hyers-Ulam stability of second order linear differential equation

$y''(x)+\beta(x)y(x)=0$ .... (3)

with boundary conditions $y(a)=y(b)=0$

where, $y\in c^2[a,b],\beta(x)\in c[a,b],-\infty<a<b<+\infty$

Definition: We say (3) has the Hyers-Ulam stability with boundary conditions

$y(a)=y(b)=0$ if there exists a positive constant K with the following property:

For every $\varepsilon>0, y\in c^2[a,b]$, if

$$\left|y''(x)+\beta(x)y(x)\right|\le\varepsilon,$$

And $y(a)=y(b)=0$, then there exists some $z\in c^2[a,b]$ satisfying

$$z''(x)+\beta(x)z(x)=0$$

And $z(a)=0=z(b)$, such that $\left|y(x)-z(x)\right|<K\varepsilon$

Theorem 1[1]. Consider the differential equation

$y''(x)+\beta(x)y(x)=0$ -------------------- (4)

With boundary conditions $y(a)=y(b)=0$

Where $y\in c^2[a,b],\beta(x)\in c[a,b],-\infty<a<b<+\infty$

If $\max\left|\beta(x)\right|<\dfrac{8}{(b-a)^2}$

then the equation (4) above has the super stability with boundary condition

$y(a)=y(b)=0$

Proof:

For every $\varepsilon>0, y\in C^2[a,b]$, if

$\left|y''(x)+\beta(x)y(x)\right|\le\varepsilon$ and $y(a)=0=y(b)$

Let M = $\max\{\left|y(x)\right|\}: x\in[a,b]$, since $y(a)=0=y(b)$, there exists $x_0\in(a,b)$

Such that $\left|y(x_0)\right|$ = M . By Taylor formula, we have

$$y(a)=y(x_0)+y'(x_0)(x_0-a)+\frac{y''(\delta)}{2!}(x_0-a)^2,$$

$$y(b)=y(x_0)+y'(x_0)(b-x_0)+\frac{y''(\eta)}{2!}(b-x_0)^2,$$

Where $\delta,\eta\in(a,b)$

Thus $\left|y''(\delta)\right|=\dfrac{2M}{(x_0-a)^2}$,

$$\left|y''(\eta)\right|=\frac{2M}{(x_0-b)^2}$$

On the case $x_0\in\left(a,\dfrac{(a+b)}{2}\right]$, we have

$$\frac{2M}{(x_0-a)^2}\ge\frac{2M}{\dfrac{(b-a)^2}{4}}=\frac{8M}{(b-a)^2}$$

On the case $x_0\in\left[\dfrac{(a+b)}{2},b\right)$ we have

$$\frac{2M}{(b-x_0)^2}\ge\frac{2M}{\dfrac{(b-a)^2}{4}}=\frac{8M}{(b-a)^2}$$

So $\max\left|y''(x)\right|\ge\dfrac{8M}{(b-a)^2}=\dfrac{8}{(b-a)^2}max\left|y(x)\right|$

Therefore $max\left|y(x)\right|\le\dfrac{(b-a)^2}{8}max\left|y''(x)\right|$

Thus

$$max\left|y(x)\right|\le\frac{(b-a)^2}{8}\left[max\left|y''(x)-\beta(x)y(x)\right|+max\left|\beta(x)\right|ma\right.$$

$$\le\frac{(b-a)^2}{8}\varepsilon+\frac{(b-a)^2}{8}\max\left|\beta(x)\right|\max\left|y(x)\right|$$

Let $k=\dfrac{(b-a)^2}{8\left(1-\dfrac{(b-a)^2}{8}\max\left|\beta(x)\right|\right)}$

Obviously, $z_0(x)=0$ is a solution of $y''(x)-\beta(x)y(x)=0$ with the Boundary conditions $y(a)=0=y(b)$.

$$\left|y(x)-z_0(x)\right|\le K\varepsilon$$

Hence the differential equation $y''(x)+\beta(x)y(x)=0$ has the super stability with boundary condition $y(a)=0=y(b)$

In (2014) the researchers J. Huang, Q. H. Aliqifiary, Y. Li established the super stability of the linear differential equations.

$$y''(x)+p(x)y'(x)+q(x)y(x)=0$$

With boundary conditions $y(a)=0=y(b)$

Where

$y\in c^2[a,b],p\in c^1[a,b],q\in c^0[a,b],-\infty<a<b<+\infty$

Then the aim of this paper is to investigate the super stability of third-order linear differential homogeneous equations by extending the work of J. Huang, Q. H. Aliqifiary and Y. Li using the standard procedures of them.

Lemma(2) .Let $y\in c^3[a,b]$ and $y(a)=0=y(b)$, then

$$\max\left|y(x)\right|\le\frac{(b-a)^3}{48}\max\left|y'''(x)\right|$$

Proof

Let $M = \max\{|y(x)| : x \in [a,b]\}$ since $y(a) = 0 = y(b)$ there exists:

$x_0 \in (a,b)$ Such that $|y(x_0)| = M$

By Taylor's formula we have

$$y(a) = y(x_0) + y'(x_0)(x_0 - a) + \frac{y''(x_0)}{2!}(x_0 - a)^2 + \frac{y'''(\delta)}{3!}(x_0 - a)^3,$$

$$y(b) = y(x_0) + y'(x_0)(b - x_0) + \frac{y''(x_0)}{2!}(b - x_0)^2 + \frac{y'''(\eta)}{3!}(b - x_0)^3,$$

$$\Rightarrow |y(a)| \le |y(x_0)| + |y'(x_0)|(x_0 - a) + \left|\frac{y''(x_0)}{2!}\right|(x_0 - a)^2 + \left|\frac{y'''(\delta)}{3!}\right|(x_0 - a)^3,$$

And $\Rightarrow |y(b)| \le |y(x_0)| + |y'(x_0)|(b - x_0) + \left|\frac{y''(x_0)}{2!}\right|(b - x_0)^2 + \left|\frac{y'''(\eta)}{3!}\right|(b - x_0)^3,$

Where $\delta, \eta \in (a,b)$

Thus $|y'''(\delta)| = \dfrac{3!M}{(x_0 - a)^3} = \dfrac{6M}{(x_0 - a)^3}$

And $|y'''(\eta)| = \dfrac{3!M}{(b - x_0)^3} = \dfrac{6M}{(b - x_0)^3}$

For the case $x_0 \in \left(a, \frac{a+b}{2}\right]$ that is $a < x_0 \le \frac{a+b}{2}$,we have

$$\frac{6M}{(x_0 - a)^3} \ge \frac{6M}{\left(\frac{a+b}{2} - a\right)^3} = \frac{6M}{\frac{(b-a)^3}{8}} = \frac{48M}{(b-a)^3}$$

And for the case $x_0 \in \left[\frac{a+b}{2}, b\right)$,that is $\frac{a+b}{2} \le x_0 < b$ we have

$$\frac{6M}{(b - x_0)^3} \ge \frac{6M}{\left(\frac{b-a}{2}\right)^3} = \frac{6M}{\frac{(b-a)^3}{8}} = \frac{48M}{(b-a)^3}$$

$$\Rightarrow \frac{6M}{(b - x_0)^3} \ge \frac{48M}{(b-a)^3}$$

Thus $\max|y'''(x)| \ge \dfrac{48M}{(b-a)^3} = \dfrac{48}{(b-a)^3}\max|y(x)|$

from $\max|y'''(x)| \ge \dfrac{48}{(b-a)^3}\max|y(x)|$

$$\Rightarrow \max|y(x)| \le \frac{(b-a)^3}{48}\max|y'''(x)|$$

Theorem 2. Consider $y'''(x) + m(x)y''(x) + p(x)y'(x) + q(x)y(x) = 0$ ----------------- (5)

with boundary conditions $y(a) = 0 = y(b)$

where $y \in c^3[a,b], m \in c^2[a,b], p \in c'[a,b], q \in c^0[a,b] - \infty < a < b < +\infty$

If $max\left|q(x) + \frac{1}{2}(p - p') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8}p^3\right| < \frac{48}{(b-a)^3}$ -------(6)

Then (5) has the super stability with boundary conditions $y(a) = 0 = y(b)$

Proof

Suppose that $y \in c^3[a,b]$ satisfies the inequality:

$\left|y'''(x) + m(x)y''(x) + p(x)y'(x) + q(x)y(x)\right| \le \varepsilon$ ------------ (7) for some $\varepsilon > 0$

Let $U(x) = y'''(x) + m(x)y''(x) + p(x)y'(x)q(x)y(x)$ --------- (8)

For all $x \in [a,b]$ and define $z(x)$ by

$$y(x) = z(x)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau} \text{ --------- (9)}$$

And by taking the first, second and third derivative of (9)

That is

$$y'(x) = \left(z' - \frac{1}{2}zp\right)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$y''(x) = \left(z'' - z'p - \frac{1}{2}zp' + \frac{1}{4}zp^2\right)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$y'''(x) = \left(z''' - \frac{1}{2}z''p - z''p - z'p' + \frac{1}{2}z'p^2 - \frac{1}{2}z'p' - \frac{1}{2}zp'' + \frac{1}{4}zpp' + \frac{1}{4}z'p^2 + \frac{1}{2}zp - \frac{1}{8}zp^3\right)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

And by substituting (9) and its first, second and third derivatives in (8) we get

$$u(x) = \Big[z''' - \frac{1}{2}z''p - z''p - z'p' + \frac{1}{2}z'p^2 - \frac{1}{2}z'p' - \frac{1}{2}zp'' + \frac{1}{4}zpp' + \frac{1}{4}z'p^2 + \frac{1}{2}pz - \frac{1}{8}zp^3 +$$

$$m\left(z'' - z'p - \frac{1}{2}zp' + \frac{1}{4}zp^2\right) + p\left(z' - \frac{1}{2}zp\right) + zq\Big]e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$= \Big[z''' - \frac{1}{2}z''p - z''p - z'p' + \frac{1}{2}z'p^2 - \frac{1}{2}z'p' - \frac{1}{2}zp'' + \frac{1}{4}zpp' + \frac{1}{4}z'p^2 + \frac{1}{2}zp - \frac{1}{8}zp^3 +$$

$$mz'' - z'mp - \frac{1}{2}zmp' + \frac{1}{4}zmp^2 + pz' - \frac{1}{2}zp^2 + qz\Big]e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$= \Big[z''' + mz'' - \frac{3}{2}z''p + z'p - z'mp - \frac{3}{2}z'p' + \frac{3}{4}z'p^2 + qz + \frac{1}{2}zp + \frac{1}{4}zpp' - \frac{1}{2}zmp' - \frac{1}{2}zp'' +$$

$$\frac{1}{4}zmp^2 - \frac{1}{2}zp^2 - \frac{1}{8}zp^3\Big]e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

$$= \Big[z''' + \left(m - \frac{3}{2}p\right)z'' + \left(p - mp - \frac{3}{2}p' + \frac{3}{4}p^2\right)z' +$$

$$\left(q+\frac{1}{2}(p-p'')+\frac{1}{4}(pp'-2mp'+mp^2-2p^2)-\frac{1}{8}p^3\right)z\ \Big]\, e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$$

Now choose $m=\frac{3}{2}p$ and $m=\dfrac{\frac{3}{4}p^2-\frac{3}{2}p'+p}{p}$

By doing this the coefficients of $z''(x)$ and $z'(x)$ will vanish.

To check whether the relation holds or not, for

$$\frac{3}{2}p=\frac{\frac{3}{4}p^2-\frac{3}{2}p'+p}{p}$$

$$\Rightarrow \frac{3}{2}p^2=\frac{3}{4}p^2-\frac{3}{2}p'+p$$

$$\Rightarrow \frac{3}{2}p'=p-\frac{3}{4}p^2$$

$$\Rightarrow \frac{3}{2}p'=p\left(1-\frac{3}{4}p\right)$$

$$\frac{3}{2}\frac{dp}{dx}=p\left(1-\frac{3}{4}p\right)$$

$$\Rightarrow \frac{dp}{p\left(1-\frac{3}{4}p\right)}=\frac{2}{3}dx$$ using partial fraction

Since $\dfrac{1}{p}+\dfrac{\frac{3}{4}}{1-\frac{3}{4}p}=\dfrac{1}{p\left(1-\frac{3}{4}p\right)}$

$$\left(\frac{1}{p}+\frac{\frac{3}{4}}{1-\frac{3}{4}p}\right)dp=\frac{2}{3}dx \quad \text{--------(*)}$$

By integrating both sides of (*) we have

$$\ln|p|-\ln\left|1-\frac{3}{4}p\right|=\frac{2}{3}x+c_1$$

$$\Rightarrow \ln\left|\frac{p}{1-\frac{3}{4}p}\right|=\frac{2}{3}x+c_1$$

$$\Rightarrow \frac{p}{1-\frac{3}{4}p}=Ce^{\frac{2}{3}x}$$

$$p=Ce^{\frac{2}{3}x}\left(1-\frac{3}{4}p\right)$$

$$= Ce^{\frac{2}{3}x} - \frac{3}{4}Cpe^{\frac{2}{3}x}$$

$$p + \frac{3}{4}Cpe^{\frac{2}{3}x} = Ce^{\frac{2}{3}x} = p\left(1 + \frac{3}{4}Ce^{\frac{2}{3}x}\right)$$

$$p = \frac{Ce^{\frac{2}{3}x}}{1 + \frac{3}{4}Ce^{\frac{2}{3}x}} \Rightarrow m = \frac{3}{2}p$$

$$m = \frac{3}{2}\left(\frac{Ce^{\frac{2}{3}x}}{1 + \frac{3}{4}Ce^{\frac{2}{3}x}}\right)$$

Then $z''' + \left(q + \frac{1}{2}(p - p'') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8}p^3\right)ze^{-\frac{1}{2}\int_a^x p(\tau)d\tau} = u(x)$

Then from the inequality (7) we get $\left|z''' + \left(q + \frac{1}{2}(p - p'') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8}p^3\right)z\right| =$

$$|u(x)|e^{\frac{1}{2}\int_a^x p(\tau)d\tau} \leq \varepsilon e^{\frac{1}{2}\int_a^x p(\tau)d\tau}$$

From the boundary condition $y(a) = 0 = y(b)$ and $y(x) = z(x)e^{-\frac{1}{2}\int_a^x p(\tau)d\tau}$

We have $z(a) = 0 = z(b)$

Define $\beta(x) = q + \frac{1}{2}(p - p'') + \frac{1}{4}(pp' - 2mp' + mp^2 - 2p^2) - \frac{1}{8}p^3$

Then $|z'''(x) + \beta z(x)| = |u(x)|e^{\frac{1}{2}\int_a^x p(\tau)d\tau} \leq \varepsilon e^{\frac{1}{2}\int_a^x p(\tau)d\tau}$

Using **lemma (2)**

$$\max|z(x)| \leq \frac{(b-a)^3}{48}\max|z'''(x)|$$

$$\leq \frac{(b-a)^3}{48}\left[\max|z'''(x) + \beta z(x)| + \max|\beta|\max|z(x)|\right]$$

$$\leq \frac{(b-a)^3}{48}\max\left\{e^{\frac{1}{2}\int_a^x p(\tau)d\tau}\right\}\varepsilon + \frac{(b-a)^3}{48}\max|\beta|\max|z(x)|$$

Since $\max e^{\frac{1}{2}\int_a^x p(\tau)d\tau} < \infty$ on the interval $[a,b]$

Hence, there exists a constant $K > 0$ such that

$|z(x)| \leq k\varepsilon$ For all $x \in [a,b]$

More over $\max e^{-\frac{1}{2}\int_a^x p(\tau)d\tau} < \infty$ on the interval $[a,b]$ which implies that there exists a constant such that $K' > 0$

$$|y(x)| = \left| z(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right|$$

$$\le \max\left\{\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right\}k\varepsilon \le k'\varepsilon$$

$$\Rightarrow |y(x)| \le k'\varepsilon$$

Then $y'''(x)+m(x)y''(x)+p(x)y'(x)+q(x)y(x)=0$ has super stability with boundary conditions $y(a)=0=y(b)$

Example

Consider the differential equation below

$$y'''+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y''+\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'+y=0 \text{ ----------- (10)}$$

With boundary conditions $y(a)=0=y(b)$

Where $y\in c^3[a,b], \frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\in c^2[a,b], \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\in c^1[a,b], 1\in c^0[a,b], -\infty<a<b<+\infty$

If $\max\left|1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\right)''+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'+\right.$

$$\left.\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\right|<\frac{48}{(b-a)^3} \text{ --------- (11)}$$

Then (10) has super stability with boundary conditions $y(a)=0=y(b)$

Suppose that $y\in c^3[a,b]$ satisfies the inequality

$$\left|y'''+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y''+\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'+y\right|\le\varepsilon \text{ ,for some } \varepsilon>0 \text{ --------- (12)}$$

Let v(x) $= y'''+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y''+\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'+y$ ------- (13)

For all $x\in[a,b]$ and define $z(x)$ by

$$y(x) = z(x)e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau} \quad \text{-----}\ (14)$$

$$y' = \left(z' - \frac{1}{2}z\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\right)e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau}$$

$$y'' = \left(z'' - z'\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right) - \frac{1}{2}z\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' + \frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2\right)e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau}$$

$$y''' = \left(z''' - \frac{1}{2}z''\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right) - z''\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right) - z'\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' + \frac{1}{2}z'\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2 -\right.$$

$$\frac{1}{2}z'\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' - \frac{1}{2}z\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'' + \frac{1}{4}z\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' +$$

$$\left.\frac{1}{4}z'\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2 + \frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)z - \frac{1}{8}z\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\right)e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau}$$

By substituting (14) and its first, second and third derivatives in (13) we get

$$\left[z''' + \left(\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right) - \frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\right)z'' +\right.$$

$$\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} - \frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right) - \frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' + \frac{3}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2\right)z'$$

$$+\left(1 + \frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right) - \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'' + \frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' - 3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'\right.$$

$$\left.\left.+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2 - 2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2 - \frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\right)z\right]e^{-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau}$$

Since z' and z''

$$v(x)=\left[z'''+\left(1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'\right.\right.$$

$$\left.\left.+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\right)z\right]e^{-\frac{1}{2}\int_a^x\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau}$$

Then from inequality (12) we get

$$\left|z'''+\left(1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'\right.\right.$$

$$\left.\left.+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3\right)z\right|$$

$$=|v(x)|\exp\left(\frac{1}{2}\int_a^x\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau\right)\le\exp\left(\frac{1}{2}\int_a^x\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau\right)\varepsilon$$

From the boundary condition $y(a)=0=y(b)$ and

$$y(x)=z(x)\exp\left(-\frac{1}{2}\int_a^x\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}d\tau\right)$$ we have $z(a)=0=z(b)$

Define:

$$\beta=1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}-\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right)+\frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'-3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)'$$

$$+\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3$$

Then $\left|z''' + \beta z(x)\right| = \left|v(x)\right| \exp\left(\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right) \le \exp\left(\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\varepsilon$

using **lemma (2)**

$$\max\left|z(x)\right| \le \frac{(b-a)^3}{48}\max\left|z'''(x)\right|$$

$$\le \frac{(b-a)^3}{48}\left[\max\left|z''' + \beta z(x)\right| + \max\left|\beta\right|\max\left|z(x)\right|\right]$$

$$\le \frac{(b-a)^3}{48}\max\left\{\exp\left(\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right\}\varepsilon + \frac{(b-a)^3}{48}\max\left|\beta\right|\max\left|z(x)\right|$$

since $\max\left\{\exp\left(\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right\} < \infty$ on the interval $[a,b]$

Hence there exists a constants $k > 0$ such that $\left|z(x)\right| \le k\varepsilon$

$$\left|y(x)\right| = \left|z(x)\exp\left(-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right|$$

$$\le \max\left\{\exp\left(-\frac{1}{2}\int_a^x \frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\,d\tau\right)\right\}k\varepsilon \le k'\varepsilon$$

$$\Rightarrow \left|y(x)\right| \le k'\varepsilon$$

Then the differential equation $y''' + \frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y'' + \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)y' + y = 0$

Has the super stability with boundary condition $y(a) = 0 = y(b)$ on closed bounded interval $[a,b]$

To show the $\max\left\{\left|1+\frac{1}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}} - \left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)''\right) + \frac{1}{4}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' - 3\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)' + \right.\right.$

$$\frac{3}{2}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-2\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^2-\frac{1}{8}\left(\frac{e^{\frac{2}{3}x}}{1+\frac{3}{4}e^{\frac{2}{3}x}}\right)^3 \Big|\Big\} < \frac{48}{(b-a)^3} ------(**)$$

To make our work simple we simplify the expression (**)

Then the simplified form of (**) is

$$\max\left\{\left|1+\frac{\frac{5}{6}e^{\frac{2}{3}x}+\frac{15}{8}\left(e^{\frac{2}{3}x}\right)^2+\frac{35}{32}\left(e^{\frac{2}{3}x}\right)^3-\frac{33}{128}\left(e^{\frac{2}{3}x}\right)^4}{\left(1+\frac{3}{2}e^{\frac{2}{3}x}\right)^4}\right|\right\} < \frac{48}{(b-a)^3} \text{ ---------- (***)}$$

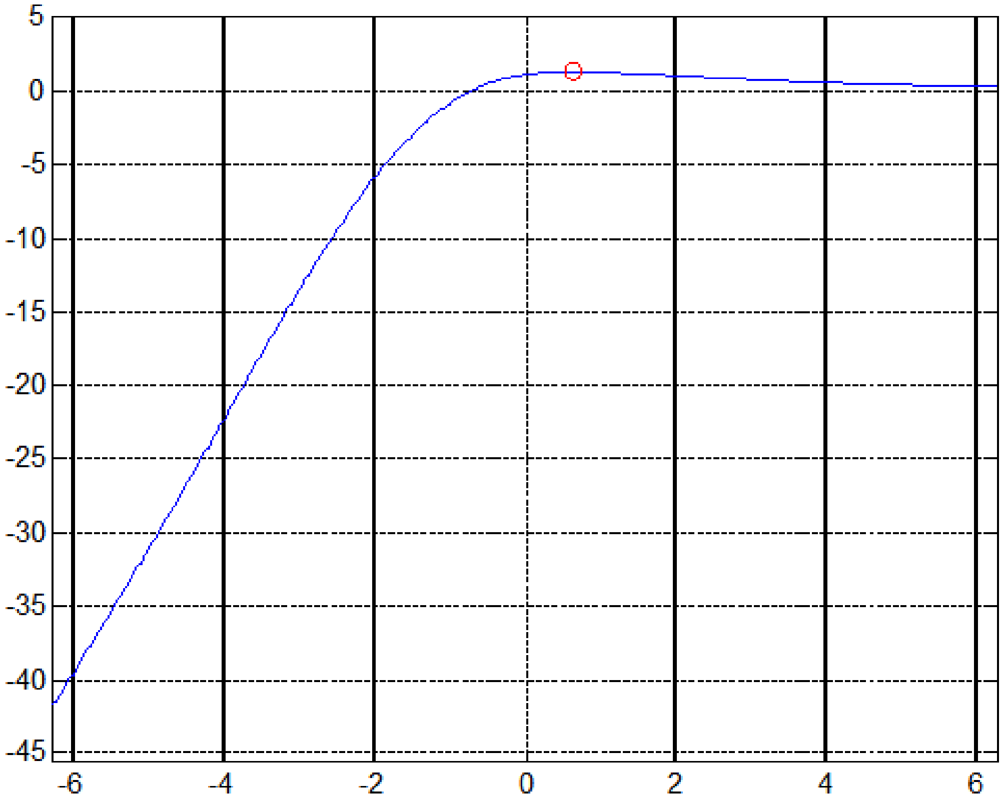

Figure:1

The graph which shows the $\max\left\{\left|1+\frac{\frac{5}{6}e^{\frac{2}{3}x}+\frac{15}{8}\left(e^{\frac{2}{3}x}\right)^2+\frac{35}{32}\left(e^{\frac{2}{3}x}\right)^3-\frac{33}{128}\left(e^{\frac{2}{3}x}\right)^4}{\left(1+\frac{3}{2}e^{\frac{2}{3}x}\right)^4}\right|\right\} < \frac{48}{(b-a)^3}$

Conclusion

In this study, the super stability of third order linear ordinary differential homogeneous equation in the form of $y'''(x)+m(x)y''(x)+p(x)y'(x)+q(x)y(x)=0$ with boundary condition was established. And the standard work of JinghaoHuang, QusuayH.Alqifiary, Yongjin Li on investigating the super stability of second order linear ordinary differential homogeneous equation with boundary condition is extended to the super stability of third order linear ordinary differential homogeneous equation with Dirchilet boundary condition.

In this thesis the super stability of third order ordinary differential homogeneous equation was established using the same procedure coming researcher can extend for super stability of higher order differential homogeneous equation.

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