Errors in the Discretized Solution of a Differential Equation

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ABSTRACT

We study the error in the derivatives of an unknown function. We construct the discretized problem. The local truncation and global errors are discussed. The solution of discretized problem is constructed. The analytical and discretized solutions are compared. The two solution graphs are described by using MATLAB software.

KEYWORDS: Differential Equations, MATLAB, Heat Equation

1. INTRODUCTION
1.1. Measuring Errors
In order to discuss the accuracy of a numerical solution, it is necessary to choose a manner of measuring that error. It may seem obvious what is meant by the error, but as we will see there are often many different ways to measure the error which can sometimes gives quite different impressions as to the accuracy of an approximate solution.

1.1.1. Errors in a Scalar Value
First we consider a problem in which the answer is a single value $z \in R$. Consider, for example, the scalar ODE

$$u'(t) = f(u(t)), u(0) = \eta \qquad (1.1)$$

and suppose we are trying to compute the solution at some particular time T, so $z = u(T)$. Denote the computed solution by

$\hat{z}$. Then the error in this computed solution is

$$E = \hat{z} - z \qquad (1.2)$$

1.1.2. Absolute Error
A natural measure of this error would be the absolute value of E,

$$|E| = |\hat{z} - z| \qquad (1.3)$$

This is called the **absolute error** in the approximation.

1.1.3. Relative Error
The error defined by $\left| \dfrac{\hat{z} - z}{z} \right|$ is called **relative error**.

1.1.4. "Big-oh" and "little-oh" notation
In discussing the rate of convergence of a numerical method we use the notation $O(\tau^p)$, the so-called "big-oh" notation. If $f(\tau)$ and $g(\tau)$ are two functions of τ then we say that $f(\tau) = O(g(\tau))$ as $\tau \to 0$.

If there is some constant C such that $\left| \dfrac{f(\tau)}{g(\tau)} \right| < C$ for all τ sufficiently small, or equivalently, if we can bound $|f(\tau)| < C|g(\tau)|$ for all τ sufficiently small.

It is also sometimes convenient to use the "little-oh" notation $f(\tau) = O(g(\tau))$ as $\tau \to 0$. This means that $\left| \dfrac{f(\tau)}{g(\tau)} \right| \to 0$ as $\tau \to 0$. This is slightly stronger than the previous statement, and means that $f(\tau)$ decays to zero faster than $g(\tau)$. If $f(\tau) = o(g(\tau))$ then $f(\tau) = O(g(\tau))$ though the converse may not be true. Saying that $f(\tau) = o(1)$ simply means that the $f(\tau) \to 0$ as $\tau \to 0$.

Examples 1.1.1
$2\tau^3 = O(\tau^2)$ as $\tau \to 0$, since $\dfrac{2\tau^3}{\tau^2} = 2\tau < 1$ for all $\tau < \dfrac{1}{2}$.

$2\tau^3 = o(\tau^2)$ as $\tau \to 0$, since $2\tau \to 0$ for all $\tau < \dfrac{1}{2}$.

$\sin(\tau) = O(\tau)$ as $\tau \to 0$, since $\sin(\tau) = \tau - \dfrac{\tau^3}{3} + \dfrac{\tau^5}{5} + ... < \tau$ for all $\tau > 0$.

$\sin(\tau) = \tau + o(\tau)$ as $\tau \to 0$, since $\dfrac{(\sin(\tau) - \tau)}{\tau} = O(\tau^2)$.

1.1.5. Taylor Expansion
Each of the function values of *u* can be expanded in a Taylor series about the point *x*, as e.g.,

$$u(x + \tau) = u(x) + \tau u'(x) + \frac{1}{2}\tau^2 u''(x) + \frac{1}{6}\tau^3 u'''(x) + O(\tau^4) \qquad (1.4)$$

$$u(x - \tau) = u(x) - \tau u'(x) + \frac{1}{2}\tau^2 u''(x) - \frac{1}{6}\tau^3 u'''(x) + O(\tau^4) \qquad (1.5)$$

1.2. Finite Difference Approximations
Our goal is to approximate solutions to differential equation, i.e. to find a function (or some discrete approximation to this function) which satisfies a given relationship between

various of its derivatives on some given region of space and/ or time along with some boundary conditions along the edges of this domain. A finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. This gives a large algebraic system of equations to be solve in place of the differential equation, something that is easily solved on a computer.

We first consider the more basic question of how we can approximate the derivatives of a known function by finite difference formulas based only on values of the function itself at discrete points. Besides providing a basis for the later development of finite difference methods for solving differential equations, this allows us to investigate several key concepts such as the **order of accuracy** of an approximation in the simplest possible setting.

Let $u(x)$ represent a function of one variable that will always be assumed to be smooth, defined bounded function over an interval containing a particular point of interest x,

1.2.1. First Derivatives
Suppose we want to approximate $u'(x)$ by a finite difference approximation based only on values of u at a finite number of points near x. One choice would be to use

$$\delta_+ u(x) = \frac{u(x+\tau) - u(x)}{\tau}$$

$$= u'(x) + \frac{1}{2}\tau u''(x) + O(\tau^2) \qquad \textbf{(1.6)}$$

for some small values of τ. It is known as **forward difference approximation**.

Another **one-sided** approximation would be

$$\delta_- u(x) = \frac{u(x) - u(x-\tau)}{\tau}$$

$$= u'(x) - \frac{1}{2}\tau u''(x) + O(\tau^2) \qquad \textbf{(1.7)}$$

for two different points. It is known as **backward difference approximation**.

Another possibility is to use the **centered difference approximation**

$$\delta_0 u(x) = \frac{u(x+\tau) - u(x-\tau)}{2\tau}$$

$$= \frac{1}{2}[\delta_+ u(x) + \delta_- u(x)]$$

$$= u'(x) + \frac{1}{6}\tau^2 u''' + O(\tau^4) \qquad \textbf{(1.8)}$$

1.2.2. Second Order Derivatives
The standard second order **centered approximation** is given by

$$\delta^2 u(x) = \frac{u(x-\tau) - 2u(x) + u(x+\tau)}{\tau^2} \qquad \textbf{(1.9)}$$

$$= u''(x) + \frac{1}{12}\tau^2 u''''(x) + O(\tau^4)$$

$$= u''(x) + O(\tau^2)$$

1.2.3. Higher Order Derivatives
Finite difference approximations to higher order derivatives can be obtained.

$$\delta_+ \delta^2 u(x) = \frac{1}{\tau^3}[u(x+2\tau) - 3u(x+\tau) + 3u(x) - u(x-\tau)] \qquad \textbf{(1.10)}$$

$$= u'''(x) + \frac{1}{2}\tau u''''(x) + O(\tau^2)$$

$$= u'''(x) + O(\tau)$$

The first one equation (1.10) is un-centered and first order accurate:

$$\delta_0 \delta_+ \delta_- u(x) = \frac{1}{2\tau^3}[u(x+2\tau) - 2u(x+\tau) + 2u(x-\tau) - u(x-2\tau)] \qquad \textbf{(1.11)}$$

$$= u'''(x) + \frac{1}{4}\tau^2 u''''(x) + O(\tau^4)$$

$$= u'''(x) + O(\tau^2)$$

This second equation (1.12) is second order accurate.

2. Comparison Of Analytical and Discretized Solution Of Heat Equation
2.1 Solutions for the Heat Equation
2.1.1. Finite Difference Method
We will derive a finite difference approximation of the following initial boundary value problem:

$$u_t = u_{xx} \text{ for } x \in (0,1), t > 0$$

$$u(0,t) = u(1,t) = 0 \text{ for } t \geq 0 \qquad \textbf{(2.1)}$$

$$u(x,0) = f(x) \text{ for } x \in (0,1)$$

Let $n \geq 0$ be a given integer, and define the grid spacing in the x-direction by

$$\Delta x = \tau = \frac{1}{(n+1)}$$

The grid points in the x-direction are given by $x_j = j\tau$ for *j=0,1,…,n+1*. Similarly, we define $t_m=mh$ for integers $m \geq 0$, where $h = \Delta t$ denotes the time step. Then, we let

v_j^m denote an approximation of $u(x_j, t_m)$. We have the following approximations

$$u_t(x,t) = \frac{u(x,t+h) - u(x,t)}{h} + O(h) \qquad \textbf{(2.2)}$$

and

$$u_{xx}(x,t) = \frac{u(x-\tau,t) - 2u(x,t) + u(x+\tau,t)}{\tau^2} + O(\tau^2) \qquad \textbf{(2.3)}$$

These approximations motivate the following scheme:

$$\frac{v_j^{m+1} - v_j^m}{h} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\tau^2} \text{ for } j=1,\dots,n, \ m \geq 0 \qquad \textbf{(2.4)}$$

By using the boundary conditions of (2.1), we have
$v_0^m = 0$ and $v_{n+1}^m = 0$, for all $m \geq 0$.

The scheme is initialized by
$v_j^0 = f(x_j)$, for j=1,...,n.

Let $r = \dfrac{h}{\tau^2}$. Then the scheme can be rewritten in a more convenient form
$$v_j^{m+1} = rv_{j-1}^m + (1-2r)v_j^m + rv_{j+1}^m, \ j=1,\dots,n, \ m \geq 0 \qquad \textbf{(2.5)}$$

When the scheme is written in this form, we observe that the values on the time level tm+1 are computed using only the values on the previous time level tm and we have to solve a tridiagonal system of linear equations.

2.1.2 Approximate Solution

The first step in our discretized problem is to derive a family of particular solutions of the following problem:

$$\frac{v_j^{m+1} - v_j^m}{h} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\tau^2} \text{ for } j=1,\dots,n, \ m \geq 0 \qquad \textbf{(2.6)}$$

with the boundary conditions
$v_0^m = 0$ and $v_{n+1}^m = 0$, for all $m \geq 0$. $\qquad \textbf{(2.7)}$

The initial data will be taken into account later. We seek particular solutions of the form
$v_j^m = X_j T_m$ for j=1,...,n, $m \geq 0$ $\qquad \textbf{(2.8)}$

Here X is a vector of n components, independent of m, while $\{T_m\}_{m \geq 0}$ is a sequence of real numbers. By inserting (2.8) into (2.6), we get
$$\frac{X_j T_{m+1} - X_j T_m}{h} = \frac{X_{j-1} T_m - 2X_j T_m + X_{j+1} T_m}{\tau^2}$$

Since we are looking only for nonzero solutions, we assume that $X_j T_m \neq 0$, and thus we obtain
$$\frac{T_{m+1} - T_m}{h T_m} = \frac{X_{j-1} - 2X_j + X_{j+1}}{\tau^2 X_j}.$$

The left-hand side only depends on m and the right-hand side only depends on j. Consequently, both expressions must be equal to a common constant, say $(-\mu)$, and we get the following two difference equations:
$$\frac{X_{j-1} - 2X_j + X_{j+1}}{\tau^2} = -\mu X_j, \text{ for } j=1,\dots,n, \qquad \textbf{(2.9)}$$

$$\frac{T_{m+1} - T_m}{h} = -\mu T_m \text{ for } m \geq 0. \qquad \textbf{(2.10)}$$

We also derive from the boundary condition (2.7) that
$$X_0 = X_{n+1} = 0 \qquad \textbf{(2.11)}$$

We first consider the equation (2.10). We define T0=1 and consider the difference equation

$$T_{m+1} = (1 - h\mu)T_m \text{ for } m \geq 0. \qquad \textbf{(2.12)}$$

Some iterations of (2.12)

$$\begin{aligned}
T_{m+1} &= (1 - h\mu)T_m \\
&= (1-h\mu)^2 T_{m-1} \\
&\quad . \\
&\quad . \\
&\quad . \\
&= (1-h\mu)^{m+1} T_0 \\
&= (1-h\mu)^{m+1}
\end{aligned}$$

Clearly indicate that the solution is
$$T_m = (1 - h\mu)^m \text{ for } m \geq 0. \qquad \textbf{(2.13)}$$

This fact is easily verified by induction on m. Next we turn our attention to the problem (2.9) with boundary condition (2.11). In fact this is equivalent to the eigenvalue problem.

Hence, we obtain that the n eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ are given by

$$\mu_k = \frac{2}{\tau^2} - \frac{2}{\tau^2}\cos(\kappa\pi\tau)$$

$$= \frac{2}{\tau^2}(1 - \cos(\kappa\pi\tau))$$

$$= \frac{2}{\tau^2} 2\sin^2\left(\frac{\kappa\pi\tau}{2}\right)$$

$$\mu_k = \frac{4}{\tau^2} 2\sin^2\left(\frac{\kappa\pi\tau}{2}\right) \text{ for } \kappa = 1,\dots,n \qquad \textbf{(2.14)}$$

and the corresponding eigenvectors
$$X_k = (X_{\kappa,1}, X_{\kappa,2}, \dots, X_{\kappa,n}) \in R^n, \ \kappa = 1,\dots,n$$

have components given by
$X_{\kappa,j} = \sin(\kappa\pi j\tau)$

$X_{\kappa,j} = \sin(\kappa\pi j x_j)$, j=1,...,n.

It can be easily verified that
$$AX_{\kappa,j} = \frac{-1}{\tau^2}X_{\kappa,j-1} + \frac{2}{\tau^2}X_{\kappa,j} - \frac{1}{\tau^2}X_{\kappa,j+1}$$

$$= -\frac{1}{\tau^2}\sin(\kappa\pi(j-1)\tau) + \frac{2}{\tau^2}\sin(\kappa\pi j\tau) - \frac{1}{\tau^2}\sin(\kappa\pi(j+1)\tau)$$

$$= -\frac{1}{\tau^2}\sin(\kappa\pi j\tau - \kappa\pi\tau)) + \frac{2}{\tau^2}\sin(\kappa\pi j\tau) - \frac{1}{\tau^2}\sin(\kappa\pi j\tau + \kappa\pi\tau)$$

$$= -\frac{1}{\tau^2}\left(\begin{array}{l}\sin(\kappa\pi j\tau)\cos(\kappa\pi\tau) - \cos(\kappa\pi j\tau)\sin(\kappa\pi\tau) - 2\sin(\kappa\pi j\tau) \\ + \sin(\kappa\pi j\tau)\cos(\kappa\pi\tau) + \cos(\kappa\pi j\tau)\sin(\kappa\pi\tau)\end{array}\right)$$

$$= \frac{2}{\tau^2}\sin(\kappa\pi j\tau)(1 - \cos(\kappa\pi\tau))$$

$$= \frac{2}{\tau^2}\sin(\kappa\pi j\tau)\left(2\sin^2\left(\frac{\kappa\pi\tau}{2}\right)\right)$$

$$= \frac{4}{\tau^2} \sin^2(\frac{\kappa\pi\tau}{2})\sin(\kappa\pi j\,\tau)$$

$$= \mu_\kappa X_{\kappa,j}$$

Hence, we obtain particular solutions $v_{\kappa,j}^m$ of the form

$$v_{\kappa,j}^m = (1 - h\mu_\kappa)^m \sin(\kappa\pi x_j) \qquad \textbf{(2.15)}$$

We have derived a family of particular solutions $\{v_\kappa\}_{\kappa=1}^n$ with values $v_{\kappa,j}^m$ at the grid point (xj,tm). Next, we observe that any linear combination of particular solutions

$$v = \sum_{\kappa=1}^n \gamma_\kappa v_\kappa \quad , (\gamma_\kappa \text{ is scalar})$$

is also a solution of (2.6) and (2.7). Finally, we determine the coefficients $\{\gamma_\kappa\}$ by using the initial condition

$$v_j^0 = f(x_j) \quad \text{, for j=1,...,n.}$$

Since $v_\kappa = X_\kappa$ at t=0 , we want to determine $\{\gamma_\kappa\}$ such that

$$\sum_{\kappa=1}^n \gamma_\kappa X_{\kappa,j} = f(x_j) \quad \text{, for j=1,...,n.} \qquad \textbf{(2.16)}$$

Hence, it follows from

$$f(x_j) = \sum_{\kappa=1}^n \gamma_\kappa \sin(\kappa\pi x_j)$$

$$\langle X_{sj}, f(x_j) \rangle_h = \sum_{\kappa=1}^n \gamma_\kappa \langle \sin\kappa\pi x_j, \sin s\pi x_j \rangle_h$$

that

$$\gamma_\kappa = 2\tau \sum_{j=1}^n f(x_j) X_{\kappa,j} \quad \text{for } \kappa = 1,...,n$$

2.1.3. Exact Solution
To find a solution of the initial-boundary value problem (2.1), assume that
$$u(x,t) = X(x)T(t) \qquad \textbf{(2.17)}$$

Using boundary conditions we get

$$X(0) = X(1) = 0$$

If we insert the (2.17) in the equation (2.1), we have
$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{T(t)} \qquad \textbf{(2.18)}$$

Now we observe that the left hand side is a function of x , while the right hand side just depends on t. Hence, both have to be equal to the same constant $-\lambda \in R$. This yields the eigenvalue problem for x
$$X'' = -\lambda X \text{, X(0) = X(1) = 0} \qquad \textbf{(2.19)}$$

Nontrivial solutions only exist for special values of λ_κ
They are so-called the eigenfunctions. In this special case we have the eigenvalues

$$\lambda_\kappa = (\kappa\pi)^2 \text{ for } \kappa = 1,2,..... \qquad \textbf{(2.20)}$$

with the eigenfunctions
$$X_\kappa(x) = \sin(\kappa\pi x) \text{ for } \kappa = 1,2,..... \qquad \textbf{(2.21)}$$

Further, the solution of $\dot{T} = -\lambda_\kappa T$ is given by $T_\kappa(t) = e^{-(\kappa\pi)^2 t}$ for $\kappa = 1,2,.....$ \qquad \textbf{(2.22)}

Finally, we use the superposition principle to a solution to the initial-boundary value problem (2.1), and we get

$$u(x,t) = \sum_{\kappa=1}^\infty c_\kappa T_\kappa(t) X_\kappa(x)$$

$$u(x,t) = \sum_{\kappa=1}^\infty c_\kappa e^{-(\kappa\pi)^2 t} \sin(\kappa\pi x)$$

The unknown coefficients can be determined from the initial condition such that

$$f(x) = \sum_{\kappa=1}^\infty c_\kappa \sin(\kappa\pi x) \qquad \textbf{(2.23)}$$

The solution of the continuous problem is given by
$$u(x,t) = \sum_{\kappa=1}^\infty c_\kappa e^{-\lambda_\kappa t} \sin(\kappa\pi x) \qquad \textbf{(2.24)}$$

where $\lambda_\kappa = (\kappa\pi)^2$ fourier coefficient

$$c_\kappa = 2\int_0^1 f(x)\sin(\kappa\pi x)dx \qquad \textbf{(2.25)}$$

2.2. Comparison of Analytical and Discretized Solution
2.2.1. We want to compare this analytical solution with the discretized solution given by
$$v_j^m = \sum_{\kappa=1}^n \gamma_\kappa (1 - h\mu_\kappa)^m \sin(\kappa\pi x_j) \qquad \textbf{(2.26)}$$

where
$$\mu_\kappa = \frac{4}{\tau^2} \sin^2\left(\frac{\kappa\pi\tau}{2}\right) \qquad \textbf{(2.27)}$$

and
$$\gamma_\kappa = 2\tau \sum_{j=1}^n f(x_j) \sin(\kappa\pi x_j) \text{ for } \kappa = 1,...,n \qquad \textbf{(2.28)}$$

In order to compare the analytical and discretized solution at a grid point (xj,tm), we define $u_j^m = u(x_j, t_m)$, i.e,

$$u_j^m = \sum_{\kappa=1}^\infty c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa\pi x_j) \qquad \textbf{(2.29)}$$

Our aim is to prove
$$v_j^m \approx u_j^m$$

under appropriate conditions on the mesh parameters τ and h. To avoid technicalities, we consider a fixed grid point (xj,tm) where $t_m \geq \bar{t}$ for $\bar{t} > 0$ independent of the mesh

parameters. Furthermore, we assume that the initial function f is smooth and satisfies the boundary conditions, i.e. f(0) =f(1) =0. Finally we assume that the mesh parameters h and τ are sufficiently small.

In order to compare u_j^m and v_j^m, we note that

$$u(x_j, t_m) = \sum_{\kappa=1}^{\infty} c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa \pi x_j)$$

$$= \sum_{\kappa=1}^{n} c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa \pi x_j) + \sum_{\kappa=n+1}^{\infty} c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa \pi x_j)$$

Here we want to show that

$$\sum_{\kappa=n+1}^{\infty} c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa \pi x_j) \approx 0 \qquad \textbf{(2.30)}$$

Since f is smooth, it is also bounded and then the Fourier coefficients c_κ are bounded for all κ.
Obviously, we have

$$\left| \sin(\kappa \pi x_j) \right| \leq 1$$

and

$$\left| \sum_{\kappa=n+1}^{\infty} c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa \pi x_j) \right| \leq \sum_{\kappa=n+1}^{\infty} |c_\kappa| \left| e^{-\lambda_\kappa t_m} \right| \underbrace{\left| \sin(\kappa \pi x_j) \right|}_{\leq 1}$$

$$\leq \max_\kappa |c_\kappa| \sum_{\kappa=n+1}^{\infty} e^{-(\kappa\pi)^2 t_m}$$

$$\leq C \sum_{\kappa=n+1}^{\infty} \left(e^{-\pi^2 \bar{t}} \right)^\kappa, t_m \geq \bar{t}$$

$$= C \left[\left(e^{-\pi^2 \bar{t}} \right)^{n+1} + \left(e^{-\pi^2 \bar{t}} \right)^{n+2} + \dots \right]$$

$$= C \left(e^{-\pi^2 \bar{t}} \right)^{n+1} \left[1 + e^{-\pi^2 \bar{t}} + e^{-2\pi^2 \bar{t}} + \dots \right]$$

$$= C \left(e^{-\pi^2 \bar{t}} \right)^{n+1} \frac{1}{1 - e^{-\pi^2 \bar{t}}}$$

≈ 0, for large value of n.
Since we have verified (2.30) it follow that

$$u_j^m \approx \sum_{\kappa=1}^{n} c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa \pi x_j) \qquad \textbf{(2.31)}$$

Now we want to compare the finite sums (2.26) and (2.31):

$$v_j^m = \sum_{\kappa=1}^{n} \gamma_\kappa (1 - h\mu_\kappa)^m \sin(\kappa \pi x_j)$$

Motivated by the derivation of the solutions, we try to compare the two sums term wise. Thus we keep κ fixed, and we want to compare

$c_\kappa e^{-\lambda_\kappa t_m} \sin(\kappa \pi x_j)$ and $\gamma_\kappa (1 - h\mu_\kappa)^m \sin(\kappa \pi x_j)$.

Since the sine part here is identical, it remains to compare the Fourier coefficients c_κ and γ_κ, and the time-dependent terms $e^{-\lambda_\kappa t_m}$ and $(1 - h\mu_\kappa)^m$.

2.2.2. Comparison of Fourier coefficient c_κ and coefficient γ_κ
We start by considering the Fourier coefficients, and note that γ_κ is a good approximation of c_κ because

$$2\tau \sum_{j=1}^{n} f(x_j) \sin(\kappa \pi x_j)$$

is the trapezoidal-rule approximation of

$$2 \int_0^1 f(x) \sin(\kappa \pi x) dx$$

In fact we have

$$\left| c_\kappa - \gamma_\kappa \right| = \left| 2 \int_0^1 f(x) \sin(\kappa \pi x) dx - 2\tau \sum_{j=1}^{n} f(x_j) \sin(\kappa \pi x_j) \right|$$

$$= \left| 2 \frac{\tau}{2} \sum_{j=1}^{n} \left[f(x_{j-1}) + f(x_j) \right] \sin(\kappa \pi x_j) - 2\tau \sum_{j=1}^{n} f(x_j) \sin(\kappa \pi x_j) \right|$$

$$= \left| \tau \sum_{j=1}^{n} \left[f(x_j - \tau) - f(x_j) \right] \sin(\kappa \pi x_j) \right|$$

$$= \left| \tau \sum_{j=1}^{n} \left[f(x_j) - \tau f'(x_j) + \frac{\tau^2}{2} f''(x_j) - + \dots - f(x_j) \right] \sin(\kappa \pi x_j) \right|$$

$$\leq \left| \sum_{j=1}^{n} \left[-\tau^2 f'(x_j) + \frac{\tau^3}{2} f''(x_j) - + \dots \right] \right| \left| \sin(\kappa \pi x_j) \right|$$

$$\leq \sum_{j=1}^{n} \left[-\tau^2 f'(x_j) + \frac{\tau^3}{2} f''(x_j) + \dots \right]$$

$= O(\tau^2)$, for f sufficiently smooth.

2.2.3 Comparison of the Terms $(1 - h\mu_\kappa)^m$ and $e^{-\lambda_\kappa t_m}$
We will compare the term $(1 - h\mu_\kappa)^m$ approximates the term $e^{-\lambda_\kappa t_m}$, we simplify the problem a bit by choosing a fixed time tm , say tm=1 and we assume that $h = \frac{\tau^2}{2}$. As a consequence, we want to compare the terms

$$\alpha_\kappa = e^{-\lambda_\kappa} \qquad \textbf{(2.32)}$$

and

$$\beta_\kappa = (1 - h\mu_\kappa)^{\frac{1}{h}} \qquad \textbf{(2.33)}$$

Since both α_κ and β_κ are very small for large values of κ, it is sufficient to compare them for small κ. In order to compare α_κ and β_κ for small values of κ, we start by recalling that

$$\sin(y) = y + O(y^3)$$

Thus, we get

$$2\sin^2\left(\kappa\pi\sqrt{\frac{h}{2}}\right) = 2\left[\kappa\pi\sqrt{\frac{h}{2}} - \frac{(\kappa\pi)^3 \left(\frac{h}{2}\right)^{\frac{3}{2}}}{3!} + \frac{(\kappa\pi)^5 \left(\frac{h}{2}\right)^{\frac{5}{2}}}{5!} - + \dots \right]^2$$

$$= 2(\kappa\pi)^2 \frac{h}{2}\left[1 - \frac{(\kappa\pi)^2\left(\frac{h}{2}\right)}{3!} + \frac{(\kappa\pi)^4\left(\frac{h}{2}\right)^2}{5!} - + \ldots\right]^2$$

$$\approx 2(\kappa\pi)^2 \frac{h}{2} \text{ , for h sufficiently small}$$
$$= \kappa^2\pi^2 h$$

By using these facts, we derive

$$\beta_\kappa = (1 - h\mu_\kappa)^{\frac{1}{h}}$$

$$= \left(1 - h\frac{4}{\tau^2}\sin^2(\frac{\kappa\pi\tau}{2})\right)^{\frac{1}{h}}$$

$$= \left(1 - 2\sin^2\left(\kappa\pi\sqrt{\frac{h}{2}}\right)\right)^{\frac{1}{h}}$$

$$\beta_\kappa = (1 - \kappa^2\pi^2 h)^{\frac{1}{h}}$$

$$= 1 + \frac{1}{h}(-\kappa^2\pi^2 h) + \frac{\frac{1}{h}\left(\frac{1}{h}-1\right)}{2!}(-\kappa^2\pi^2 h)^2 + \ldots + (-\kappa^2\pi^2 h)^{\frac{1}{h}}$$

$$= 1 - \kappa^2\pi^2 + \frac{1-h}{2}(-\kappa^2\pi^2)^2 + \ldots + (-\kappa^2\pi^2 h)^{\frac{1}{h}}$$

$$\approx e^{-\kappa^2\pi^2} \text{ , for h sufficiently small}$$
$$= e^{-\lambda_\kappa}$$
$$\beta_\kappa \approx \alpha_\kappa$$

This shows that also the time dependent term $e^{-\lambda_\kappa t_m}$ is well approximated by its discrete $(1 - h\mu_\kappa)^m$.

2.3 Consistency
Lemma 2.3.1.
The finite difference scheme (2.6) is consistent of order (2,1).
Proof:
Local truncation error of the finite difference scheme (2.6) is given by

$$l_j^m = \frac{u_j^{m+1} - u_j^m}{h} - \frac{1}{\tau^2}\left(u_{j-1}^m - 2u_j^m + u_{j+1}^m\right)$$

$$hl_j^m = u_j^{m+1} - u_j^m - \frac{h}{\tau^2}\left(u_{j-1}^m - 2u_j^m + u_{j+1}^m\right)$$

$$= u(x_j,t_m) + hu_t(x_j,t_m) + \frac{h^2}{2}u_{tt}(x_j,t_m) + \frac{h^3}{6}u_{ttt}(x_j,t_m) + \ldots$$

$$- u(x_j,t_m) - \frac{h}{\tau^2}\left[\begin{array}{l} u(x_j,t_m) - \tau u_x(x_j,t_m) + \frac{\tau^2}{2}u_{xx}(x_j,t_m) - \frac{\tau^3}{6}u_{xxx}(x_j,t_m) \\ + \frac{\tau^4}{24}u_{xxxx}(x_j,t_m) - + \ldots - 2u(x_j,t_m) + u(x_j,t_m) + \tau u_x(x_j,t_m) \\ + \frac{\tau^2}{2}u_{xx}(x_j,t_m) + \frac{\tau^3}{6}u_{xxx}(x_j,t_m) + \frac{\tau^4}{24}u_{xxxx}(x_j,t_m) + \ldots \end{array}\right]$$

$$= hu_t(x_j,t_m) + \frac{h^2}{2}u_{tt}(x_j,t_m) + \ldots$$
$$- \frac{h}{\tau^2}\left[\tau^2 u_{xx}(x_j,t_m) + \frac{\tau^4}{12}u_{xxxx}(x_j,t_m) - \ldots\right]$$

$$l_j^m = u_t(x_j,t_m) + \frac{h}{2}u_{tt}(x_j,t_m) + \ldots - u_{xx}(x_j,t_m) - \frac{\tau^2}{12}u_{xxxx}(x_j,t_m) - \ldots$$

$$= u_t(x_j,t_m) - u_{xx}(x_j,t_m) + \frac{h}{2}u_{tt}(x_j,t_m) - \frac{\tau^2}{12}u_{xxxx}(x_j,t_m) - \ldots$$

$$= \frac{h}{2}u_{tt}(x_j,t_m) - \frac{\tau^2}{12}u_{xxxx}(x_j,t_m) + \ldots$$

$$= O(\tau^2, h)$$

$\therefore$ The finite difference scheme (2.6) is consistent of order (2,1).

Example 2.3.1
Solve the **IVP**

$$u_t = u_{xx} \text{ for } x \in (0,1) \text{ , } t > 0$$
$$u(0,t) = u(1,t) = 0 \text{ for } t \geq 0 \qquad (2.34)$$

$$u(x,0) = f(x) \text{ for } x \in (0,1).$$

The exact solution is given by
$$u = -e^{-\pi^2 t}\sin\pi x$$

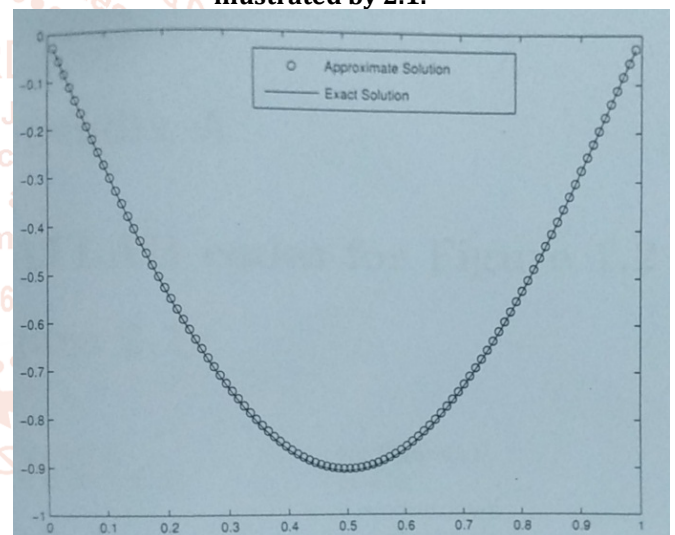**Approximate solution and exact solution are illustrated by 2.1.**

Figure2.1: Comparison of Exact and Approximate Solutions

MATLAB codes for Figure 2.1

```
N=100 ; x=zeros (N,1) ;

g= zeros (N,1) ;

v=zeros (N,N) ;

%delta x=h, delta t=p

%f (x)=cos (2*pi*x) ;
H=1/(N+1) ;
P=1/(N+1) ;

for j = 1:N
 x ( j )=j*h ;

end
%for m=1:N
% t ( m )=m*p ;
```

```
%end
%gamma k
for k=1:N

 sum=0 ;
 for j=1:N
 term=2*h*sin ( 2*pi*x ( j ))*sin (k*pi*x ( j ));
 sum=sum+term ;
 end
 g ( k )=sum ;
end
%vk
for k=1 : N
 for j = 1 : N
 v ( k , j )=p*(4/h^2)*(sin(k*pi*h/2))^2)*sin(k*pi*x(j)) ;

 end
 v ( k , : ) ;

end
sum 1 =0 ;
for k= 1 : N
```

```
term1=g ( k ) . *v ( k , :) ;
 sum1=sum1+term1 ;
 s =sum1 ;
 end

%%%Approximate solution
plot ( x , s , ' – ') ; hold on
%%%%%%Exact solution
Plot ( x ,-exp (-*pi^2*0.01 )*sin(pi*x) ) ;
```

3. CONCLUSION
The aim of this research paper describe the Errors between Analytical solution and Discretized solution of Differential equations.

4. REFERENCES
[1] R. J. LeVeque, Finite difference methods for differential equations, Washington, 2006.

[2] J. W. Thomas, Numerical partial differential equation: Finite difference methods, Springer-Verlag, New York, 1995.