A linear homogeneous recurrence relation of degree \( k \) with constant coefficients is a recurrence. A recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial conditions are given; each further term of the sequence or array is defined as a function of the preceding terms.

### 1. INTRODUCTION
A recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s). The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. If we denote the \( n \)th term in the sequence by \( x_n \), such a recurrence relation is of the form \( x_{n+1} = f(x_n) \) for some function \( f \). One such example is \( x_{n+1} = 2x_n - x_{n-2} \). A recurrence relation can also be higher order, where the term \( x_{n+1} \) could depend not only on the previous term \( x_{n-1} \) but also on earlier terms such as \( x_{n-1}, x_{n-2}, \ldots \).

### 3. Methodology
Linear Homogeneous Recurrence Relations Definition: A linear homogeneous recurrence relation of degree \( k \) with constant coefficients is a recurrence relation of the form

\[
\begin{align*}
A_1 x_n + A_2 x_{n-1} + \cdots + A_{k-1} x_{n-k} + A_k x_{n-k} &= 0 \\
\end{align*}
\]

\( A_1, A_2, \ldots, A_k \) are real numbers, \( A_k \neq 0 \) it is linear because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of \( n \). It is homogeneous because no terms occur that are not multiples of the \( a_j \)s. Each coefficient is a constant. The degree is \( k \) because a \( n \) is expressed in terms of the previous \( k \) terms of the sequence. By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the \( k \) initial conditions \( a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1} \).

For example, the recurrence relation \( x_{n+1} = x_n + x_{n-1} \) can generate the Fibonacci numbers. To generate sequence based on a recurrence relation, one must start with some initial values.

A linear homogeneous recurrence of order \( k \) is expressed this way: \( A_0 x_n + A_1 x_{n-1} + A_2 x_{n-2} + \cdots + A_{k-1} x_{n-k} + A_k x_{n-k} = 0 \)

When its constant coefficients are in arithmetic, respective geometric progression. Rather surprising, when the coefficients are in arithmetic progression, the solution is a sequence of certain generalized Fibonacci numbers, but not of usual Fibonacci numbers, while if they are in geometric progression the solution is again a geometric progression, with different ratio. Recurrence relations can be divided into two: Linear and Non-Linear. Linear homogeneous recurrence relations with constant coefficients; Solving linear homogeneous recurrence relations with constant coefficients; Solving linear homogeneous recurrence relations with constant coefficients of degree two and degree three; Generating functions; Using generating functions to solve recurrence relations. Some recurrence relations are solvable using algebraic techniques, but they're often tricky, they require some math many of you don’t know, and it still won’t work for many relations.

For a first order recursion \( x_{n+1} = f(x_n) \), one just needs to start with an initial value \( x_0 \) and can generate all remaining terms using the recurrence relation. For a second order recursion \( x_{n+1} = f(x_n, x_{n-1}) \), one needs to start with two values \( x_0 \) and \( x_1 \). Higher order recurrence relations require correspondingly more initial values.

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\]

### 2. Objectives

The aim of this paper is to solve the linear recurrence relation

\[
x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_{n-1} x_1 + a_n x_0; n = 0, 1, 2, \ldots,
\]

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a_n x - c_1 x_{n-1} - c_2 x_{n-2} - \cdots - c_k x_{n-k} = 0
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\( c_1, c_2, \ldots, c_k \) are real numbers, and \( c_k \neq 0 \). It is linear because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of \( n \). It is homogeneous because no terms occur that are not multiples of the \( x_j \)s. Each coefficient is a constant. The degree is \( k \) because \( x_n \) is expressed in terms of the previous \( k \) terms of the sequence. By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the \( k \) initial conditions \( a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1} \).

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A linear homogeneous recurrence of order \( k \) is expressed this way:

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A_0 x_n + A_1 x_{n-1} + A_2 x_{n-2} + \cdots + A_{k-1} x_{n-k} + A_k x_{n-k} = 0
\]
Theorem 1: Let \( c_1 \) and \( c_2 \) be real numbers. Suppose that \( r_2 - c_1 r - c_2 = 0 \) has two distinct roots \( r_1 \) and \( r_2 \). Then the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if for \( n = 0, 1, 2, \ldots \), where \( \alpha_1 \) and \( \alpha_2 \) are constants.

Example: What is the solution to the recurrence relation

\[
\begin{align*}
x_1 &= 1, 2, \ldots, \\
x_2 &= 1, 2, \ldots \\
\end{align*}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are constants.

Theorem 2: Let \( c_1 \) and \( c_2 \) be real numbers with \( c_2 \neq 0 \). Suppose that \( r_2 - c_1 r - c_2 = 0 \) has one repeated root \( r_0 \). Then the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if \( a_n = \alpha_1 r_0^n + \alpha_2 (r_0^n)^2 \), for some constants \( \alpha_1 \) and \( \alpha_2 \).

Example: What is the solution to the recurrence relation

\[
\begin{align*}
x_1 &= 0, 1, 2, \ldots, \\
x_2 &= 0, 1, 2, \ldots \\
\end{align*}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are constants.

Theorem 3: Let \( c_1, c_2, \ldots, c_k \) be real coefficients of any degree. The characteristic equation \( r^k - c_1 r^{k-1} - \cdots - c_k = 0 \) has distinct roots. Then the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if for \( n = 0, 1, 2, \ldots \), where \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are constants.

Example: What is the solution to the recurrence relation

\[
\begin{align*}
x_1 &= 2, 2, 2, \ldots, \\
x_2 &= 2, 2, 2, \ldots \\
\end{align*}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are constants.