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On Fully Indecomposable Quaternion Doubly Stochastic Matrices

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ABSTRACT

In traditional years, fully indecomposable matrices have played an vital part in various research topics. For example, they have been used in establishing a necessary condition for a matrix to have a positive inverse also, in the case of simultaneously row and column scaling sub-ordinate to the unitarily invariant norms, the minimal condition number diagonalizable, sub-stochastic matrices , Kronecker products is achieved for fully indecomposable matrices. In the existence of diagonal matrices D1 and D2 , with strictly positive diagonal elements, such that D1 AD2 is quaternion doubly stochastic, is established for an nXn non-negative fully indecomposable matrix A. In a related scaling for fully indecomposable non-negative rectangular matrices is also discussed.

1. INTRODUCTION

In traditional years, fully indecomposable matrices have played an vital part in various research topics. For example, they have been used in establishing a necessary condition for a matrix to have a positive inverse also, in the case of simultaneously row and column scaling sub-ordinate to the unitarily invariant norms, the minimal condition number diagonalizable, sub-stochastic matrices , Kronecker products is achieved for fully indecomposable matrices. In the existence of diagonal matrices $D_1$and D_2, with strictly positive diagonal elements, such that D_1AD_2 is quaternion doubly stochastic, is established for an $n\times n$ non-negative fully indecomposable matrix A. In a related scaling for fully indecomposable non-negative rectangular matrices is also discussed.

If A is a linear operator on $H^{n\times n}$, we will denote by $\|A\|$ the operator (bound) norm of A defined as $\|A\| = \sup\|Ax\|, \|x\| = 1$. The positive operator $(A^*A)^{1/2}$ by $S^*(A)$ the vector whose co-ordinates are the singular values of A, are arranged as $S_1^*(A) \ge S_2^*(A) \ge \ldots \ge S_n^*(A)$.

$$\|A\| = \|\|A\|\| = S_1^*(A)$$

Now, if U,V are unitarily operator on $H^{n\times n}$,
then $|UAV| = V^*|A|V$ and hence.

$\|A\| = \|UAV\|$ for all unitarily operators U,V. on the space $H(n)$ of $n\times n$ quaternion doubly stochastic matrix. In this paper, some new results and characterisations for fully indecomposable matrices.

Definition 1.1 Fully Indecomposable matrices Notation

Let $A = [a_{ij}]$ be an $n\times n$ quaternion doubly stochastic matrix. If $a_{ij} > 0$, for each I and j, then we write $A >> 0$, if $a_{ij} \ge 0$, then we write $A \ge 0$, if $A \ge 0$, but $A \ne 0$, then we write $A > 0$.

PRELIMINARY RESULTS AND DEFINITIONS

Definition 1.2

An $n\times n$ quaternion doubly stochastic matrix A is indecomposable (irreducible) if no permutation matrix P exists such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

Where A_{11} and A_{22} are square non-vacuous submatrices, otherwise A is decomposable (irreducible)

Definition 1.3

If A is said to be fully indecomposable if there exists no-permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

Where A_{11} and A_{22} are square non-vacuous submatrices otherwise A is partly decomposable. Obviously, every fully indecomposable matrix is indecomposable, and every decomposable matrix is partly decomposable.

Corollary 1.4

Let A,B be a quaternion doubly stochastic matrices. Then there exists a unitary matrix U such that $|A^*B| \le \frac{1}{2}U(AA^* + BB^*)U^*$

Corollary 1.5
Let A,B be any two matrices. Then $\left\|\left|A^*B\right|\right\| \le \frac{1}{2}\left\|\left|AA^* + BB^*\right|\right\|$ for every unitarily invariant norm.

Corollary 1.6
Let A,B be an fully indecomposable quaternion doubly stochastic matrices then there exists a permutation matrices P^* and Q^* such that $\left|A^*B\right| \le \frac{1}{2}P^*(AA^* + BB^*)Q^*$.

Proposition 1.7
If a matrix is quaternion doubly stochastic, then the matrix is square.

Proof
Let $A = (a_{ij})$ be an $m \times n$ matrix such that $\sum_{i=1}^{n} a_{ij} = 1$, $1 \le j \le m$, and $\sum_{j=1}^{m} a_{ij} = 1$, $1 \le i \le n$, summing these equations we get the sum of all entries from A in two Different ways $\sum_{j=1}^{m}(a_{1j} + a_{2j} + ... + a_{nj}) = n$, $\sum_{i=1}^{n}(a_{i1} + a_{i2} + ... + a_{im}) = m$,. Hence $n = m$ similarly.

Remark 1.8
- If a hermitian matrix is quaternion doubly stochastic, then the matrix is square.
- If a symmetric matrix is quaternion doubly stochastic, then the matrix is square.

Proposition 1.9
The permanent of quaternion doubly stochastic matrix is positive.

Proof:
Since all entries of $A \in \Omega_n$ are non-negative, the permanent of A cannot be less than zero. Let's suppose $per(A) = 0$ then by using (Frobenius Kong Theorem) there exists permutation matrices P and Q with $PAQ = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, and let 0 be the zero $h \times k$ matrix $h + k = n + 1$.

Let $S(A)$ be the sum of all entries in the matrix A. Applying this summation to PAQ matrix;

$n = S(PAQ) \ge S(X) + S(Z)$. If 0 is an $h \times k$ matrix, then all the non-zero entries in the first k-columns are included in B, therefore $S(X) = k$. In the same way for Z, hence $S(Z) = h$. Therefore $h \le S(x) + S(z) = k = h$. By our supposition $n = k + h - 1$, which is a contradiction. Hence, the hypothesis is false. Therefore $per(A) > 0$.

Theorem 1.10
If $A \in \Omega_n$, then $A = \sum_{i=1}^{S} \alpha_i P_i$, where P_i's are permutation Matrices and α_i are non-negative numbers with $\sum_{i=1}^{S} \alpha_i = 1$.

In other words, every quaternion doubly stochastic matrix is a convex combination of permutation matrices.

Some of the additional elementary properties of quaternion doubly stochastic matrices are
- If $A \in \Omega_n$ and P,Q are permutation matrices, then $PAQ \in \Omega_n$
- The product of two quaternion doubly stochastic matrices of order n is also a quatrnion doubly stochastic matrix of order n.
- The product of two quaternion hermitian doubly stochastic matrices of order n is also a quaternion hermitian doubly stochastic matrix of order n.
- The product of two quaternion symmetric doubly stochastic matrices of order n is also a quaternion symmetric doubly stochastic matrix of order n.
- If $A, B \in \Omega_n$, and $\alpha \in (0,1)$, then the convex combination $\alpha A + (1-\alpha)B$ is in Ω_n

Bruladi (1996) showed that if $A \in \Omega_n$, then $per(AA^T) = per(A^2)$ if and only if A is permutation matrix. Recall that a convex hull is the set of convex combinations $\left\{\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_k x_k / k \ge 1, x_i \in H^n, \lambda_i x_i \ge 0, \text{and} \sum_{i=1}^{k} \lambda_{i=1}\right\}$

2. DIAGONALISABLE QUATERNION DOUBLY STOCHASTIC MATRICES

2.1 Introduction:
A matrix A is said to be diagonalizable if it is similar to a diagonal matrix. i.e., if there exists an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$. This is equivalent to saying that there are n linearly independent vectors in H^n that are the eigen vectors for A. The condition number of an invertible matrix S is defined as $cond(S) = \|S\|\|S^{-1}\|$

Note that $cond(S) \ge 1$ and $cond(S) = 1$ if and only if S is scalar multiple of a unitary matrix.

Theorem 2.2
Let $A = SDS^{-1}$, where D is a diagonal matrix and S an invertible matrix.Then, for any matrix B , $S(\sigma(B), \sigma(A)) \le cond(S)\|A - B\|$

Proof:
$\|A\| = \|SD^{-1}S^{-1}D\| \le con(S)\|D^{-1}\| \le con(S)/1$

$\|A\| \le 1$

[D is an Diagonal matrix lead diagonal entries are 1]

Theorem 2.3
If A,B are quaternion doubly sub-stochastic matrices with $\|AB\| \le 1$, then $\|A^*B^*\| \le 1$ for $0 \le S \le 1$.

Proof:

$$\|AB\| \le 1 \Rightarrow \|AB^2A\| \le 1 \Rightarrow AB^2A \le I$$
$$\Rightarrow B^2 \le A^{-2} \Rightarrow A^2B^{2S}A^S \le I$$
$$\Rightarrow \|A^SB^{2S}A^S\| \le 1 \Rightarrow \|A^SB^S\| \le 1$$

Theorem 2.4

If A,B are quaternion doubly sub-stochastic positive matrices with $\lambda_1(AB) \le 1$, then $\lambda_1(A^SB^S) \le 1$ for $0 \le S \le 1$.

Proof:

We can assume that $A > 0$, we then

$$\lambda_1(AB) \le 1 \Rightarrow \lambda_1(A^{1/2}BA^{1/2}) \le 1 \Rightarrow A^{1/2}BA^{1/2} \le 1$$
$$\Rightarrow B \le A^{-1} \Rightarrow B^S \le A^{-S} \Rightarrow A^{S/2}BA^{S/2} \le 1$$
$$\Rightarrow \lambda_1(A^{S/2}B^SA^{S/2}) \le 1$$
$$\Rightarrow \lambda_1(A^SB^S) \le 1$$

This proves the theorem.

3. Schwarz Inequalities

Let A,B be an quaternion doubly stochastic matrices, and r is any positive real number. Then we say that,

$$\left\|\left|A^*B\right|^r\right\|^2 \le \left\|(AA^*)^r\right\| \left\|(BB^*)^r\right\| \qquad 4.4.1$$

For every unitarily invariant norm. then choice

$r = 1/2$ gives the inequality

$$\left\|\left|A^*B\right|^{1/2}\right\|^2 \le \|A\| \|B\| \qquad 4.4.2$$

While the choice r=1,

$$\left\|A^*B\right\|^2 \le \left\|AA^*\right\| \left\|BB^*\right\| \qquad 4.4.3$$

4. KRONECKER PRODUCTS ON QUATERION DOUBLY STOCHASTIC MATRICES:

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are square matrices of order n and m respectively. Then their Kronecker product $A \otimes B$ is s square quaternion doubly stochastic matrix of order nm, where, if (i,k) denotes $m(i-1)+k$, $1 \le i \le n$, $1 \le k \le m$, then , $[A \otimes B]_{(i,k)(j,l)} = a_{ij} - b_{kl}$. It is noted here that, $A, B \ge 0$, $A \otimes B > 0$ if and only if $A > 0$ and $B > 0$.-1.1.In this section, A and B will be taken to be defines as above.

Remark 4.1

Since $(A \otimes B)^T = A^T \otimes B^T$ (t is a positive integer) using 1.1 $(A \times B)^T > 0$ if and only if $A^T > 0$ and $B^T > 0$. Hence $A \otimes B$ is primitive if and only if both A and B primitive quaternion doubly stochastic matrix.

$$\gamma(A \otimes b) = \max(\gamma(A), \gamma(B))$$

5. QUATERNION DOUBLY STOCHASTIC MATRICES:

Let A and B are in $H^{n\times n}$ quaternion doubly stochastic matrices.

A matrix that is its own inverse is said to be a involution. i.e., $AA^{-1} = A^{-1}A = I$

MAIN RESULTS

Theorem 5.1

If A and B are $n \times n$ fully indecomposable quaternion doubly stochastic matrices, then their product AB is also fully indecomposable quaternion doubly stochastic matrices.

Proof:

Suppose AB is not fully indecomposable quaternion doubly stochastic matrices, then there exists permutation matrices P and Q such that

$$PABQ = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$$

Where U and W are $r \times r$ and $(n-r)\times(n-r)$ non-vacuous sub matrices, respectively, So $PABQ$ is decomposable. But PA and BQ are fully indecomposable, So $PABQ$ is decomposable by lemma(2). Which leads to a contradiction.

Corollary 5.2

If $A \ge 0$ is $n \times n$ non-singular fully indecomposable quaternion doubly stochastic matrix, then A^{-1} is fully indecomposable quaternion doubly stochastic matrix but not non-negative.

Proof:

Suppose that A^{-1} is not fully indecomposable quaternion doubly stochastic matrix, then there exists permutation matrices p and Q such that

$$PA^{-1}Q = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

Where B_{11} and B_{22} are square non-vacuous sub matrices using the same partition Let

$$Q^TAP^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

On multiplying Q^TAP^T by $PA^{-1}Q$. The relations $A_{11}B_{11} = I$, $A_{21}B_{11} = 0$ are obtained and so B_{11} is non-singular, thus $A_{21} = 0$, which contradicts the hypothesis that A is fully indecomposable quaternion doubly stochastic matrices. If A^{-1} were non-negative, then from (**theorem 5.1**) AA^{-1} would be fully indecomposable, but this is, of course impossible.

Theorem 5.3

Let A, B, X be are quaternion doubly stochastic matrices. Then for, every unitarily invariant norm, $\left\|A^*XB\right\|^2 \le \left\|AA^*X\right\| \left\|XBB^*\right\|$

Proof

First assume that X is a positive matrix, then

$$\left\|A^*XB\right\|^2 \le \left\|A^*X^{1/2}X^{1/2}B\right\|^2 = \left\|\left(X^{1/2}A\right)^* (X^{1/2}B)^*\right\|^2$$

$$\le \left\|X^{1/2}AA*X^{1/2}\right\|\left\|X^{1/2}BB^{*}X^{1/2}\right\|$$

Using the inequality x_{max} Now use the proposition,

$$\left\|A^{*}XB\right\|^{2} \le \left\|AA*X\right\|\left\|XBB^{*}\right\|$$

This proves the theorem in this special case.

Let X be an quaternion doubly stochastic matrix and let $X = UP$ be its polar decomposition. Then by unitary invariance

$$\left\|A^{*}XB\right\| = \left\|A^{*}UPB\right\| = \left\|U^{*}A^{*}UPB\right\|$$

$$\left\|AA^{*}X\right\| = \left\|AA^{*}UP\right\| = \left\|U^{*}AA^{*}UP\right\|$$

$$\left\|XBB^{*}\right\| = \left\|UPBB^{*}\right\| = \left\|PBB^{*}\right\|$$

So, the general theorem follows by applying the special case to the triple $U^{*}AU, B, P$

6. PRODUCT OF TWO QUATERNION FULLY INDECOMPOSABLE DOUBLY STOCHASTIC MATRICES

Prove that the product of two fully indecomposable quaternion doubly stochastic matrix is also a fully indecomposable quaternion doubly stochastic matrix.

A. Let A, B be an fully indecomposable quaternion doubly stochastic matrices

$0 \le b_{kj} \le 1 \ \forall k,\ j \in \{1,2,\ldots,n\}$, $\Rightarrow 0 \le a_{ik} b_{kj} \le a_{ik} \ \forall k$, $i \in \{1,2,\ldots,n\}$

$$0 \le \sum_{k=1}^{n} a_{ik} b_{kj} \le \sum_{k=1}^{n} a_{ik}, \quad 0 \le \sum_{K=1}^{n} D_1 a_{iK} b_{Kj} D_2 \le \sum_{K=1}^{n} a_{iK}$$

Where D_1 and D_2 are diagonal matrices.

$0 \le C_{ij} \le 1$ [$\because$ A and B is an quaternion doubly stochastic matrices]

B.
$$\sum_{P=1}^{m}\left(\sum_{K=1}^{n} a_{iK} b_{KP}\right) = \sum_{K=1}^{n}\left(\sum_{P=1}^{m} a_{iK} b_{KP}\right) = \sum_{P=1}^{n}\left(\sum_{K=1}^{n} a_{iP} b_{PK}\right)$$

$$= \sum_{P=1}^{n}\left(D_1 a_{iP} \sum_{P=1}^{n} b_{PK} D_2\right)$$

But $\sum_{P=1}^{n} b_{PK}$ is the sum of elements in one row of a matrix B and its equal to 1,

$$\Rightarrow \sum_{P=1}^{n} a_{iP}.1 = \sum_{P=1}^{n} a_{iP}$$

$\sum_{P=1}^{n} a_{iP}$ is the sum of elements in one row of a matrix A equal to 1.

C. Let A,B be an quaternion doubly stochastic matrix. $C = A.B = C_{ij} = (A_{ij}).(B_{ij})$

$$C_{ij} = \sum_{K=1}^{n} a_{iK} b_{Kj},\ \forall\ i, j = 1,2,\ldots,n$$

$$\sum_{i=1}^{n} C_{ij} = \sum_{i=1}^{n}\left(\sum_{K=1}^{n} a_{iK} b_{Kj}\right) = \sum_{K=1}^{n}\left(\sum_{i=1}^{n} a_{iK} b_{Kj}\right) = \sum_{K=1}^{n}\left(\sum_{i=1}^{n} b_{Kj} a_{iK}\right)$$

$$= \sum_{K=1}^{n}\left(D_1' b_{Kj}\left(\sum_{i=1}^{n} a_{iK}\right) D_2'\right) = \sum_{K=1}^{n}\left(D_1' b_{Kj} 1 D_2'\right) = \sum_{K=1}^{n}\left(D_1' b_{Kj} D_2'\right) = 1 W$$

here D_1' and D_2' are diagonal matrices.

Theorem 6.1

Prove that the largest eigen value of a quaternion doubly stochastic matrix is 1.

Proof

The largest eigen value of a quaternion doubly stochastic matrix (i.e., a matrix whose entries are positive and whose rows add up to 1)is 1.

The eigen value 1 is obtained now suppose $Ax = \lambda x$ for some $\lambda > 1$ .Since the rows of A are non-negative & sum to 1, each element of vector Ax is a convex combination of components x, which can be no greater than x , the largest component of x.

On the other hand, at least one element of λx is greater than x_{max} which proves that $\lambda > 1$ is impossible.

Conclusion

In this article we discuss about fully indecomposable quaternion doubly stochastic matrices, diagonalizable, Kronecker product, unitarily invariant norm, condition number, permutation matrices are also discussed.

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