Existence of Extremal Solutions of Second Order Initial Value Problems

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ABSTRACT

In this paper existence of extremal solutions of second order initial value problems with discontinuous right hand side is obtained under certain monotonicity conditions and without assuming the existence of upper and lower solutions. Two basic differential inequalities corresponding to these initial value problems are obtained in the form of extremal solutions. And also we prove uniqueness of solutions of given initial value problems under certain conditions.

Keywords: *Complete lattice, Tarski fixed point theorem, isotone increasing, minimal and maximal solutions.*

1. INTRODUCTION

In [1], B C Dhage ,G P Patil established the existence of extremal solutions of the nonlinear two point boundary value problems and in [2] , B C Dhage established the existence of weak maximal and minimal solutions of the nonlinear two point boundary value problems with discontinuous functions on the right hand side. We use the mechanism of [1]and [2] ,to develop the results for second order initial value problems.

2. Second order initial value problems

Let R denote the real line and R^+, the set of all non negative real numbers. Suppose I_1= [0, A] is a closed and bounded interval in R. In this paper we shall establish the existence of maximal and minimal solutions for the second order initial value problems of the type

$x^{11} = f(t,x,x^1)$ *a.a.* $t \in I_1$ = [0,A], A>o with $x(0)=a$, $x^1(0)=b$ (2.1)

where $f: I_1 \mathrm{x} R \mathrm{x} R \longrightarrow R$ is a function and $x^1 = \dfrac{dx}{dt},\ x^{11} = \dfrac{d^2x}{dt^2}$.

By a solution x of the IVP (2.1), we mean a function $x: I_1 \longrightarrow R$ whose first derivative exists and is absolutely continuous on I_1, satisfying (2.1).

Integrating (2.1), we find that

$$x^1(t) = b + \int_0^t f(s, x(s), x^1(s))ds \text{ and}$$

$$x(t) = a + bt + \int_0^t k(t,s)\ f(s, x(s), x^1(s))\ ds \quad (2.2)$$

where the kernel $k(t,s)=t-s$ and

$$k(t,s) \in C(\Omega, I_1)\ wherein\ \Omega = \{(t,s): 0 \le s \le t \le A\}.$$

To prove the main existence result we need the following preliminaries.

Let C(I_1,R) denote the space of continuous real valued functions on I_1, AC(I_1,R) the space of all absolutely continuous functions on I_1 , M(I_1,R) the space of all measurable real valued functions on I_1 and B(I_1,R), the space of all bounded real valued functions on I_1 . By BM(I_1,R),we mean the space of bounded and measurable real valued functions on I_1, where I_1 is given interval . We define an order relation $\le$ in BM(I_1,R) by x,y$\in$ BM(I_1,R), then x$\le$y if and only if $x(t) \le y(t)$ and $x^1(t) \le y^1(t)$ for all t$\in$ I_1.

A set S in BM(I_1, R) is a complete lattice w.r.t $\le$ if supremum and infimum of every sub set of S exists in S.

Definition 2.1 A mapping T: BM(I_1,R) $\longrightarrow$ BM(I_1,R) is said to be isotone increasing
if $x,y \in$ BM(I_1,R) with $x \le y$ implies T$x \le$ Ty.

The following fixed point theorem due to Tarski [7] will be used in proving the existence of extremal solutions of second order initial value problems.

Theorem 2.2 Let E be a nonempty set and let T_1: E $\longrightarrow$ E be a mapping such that
(i) (E, $\le$) is a complete lattice
(ii) T_1 is isotone increasing and
(iii) F= {u$\in$ E / T_1 u=u}.
Then F is non empty *and* $(F,\le)$ is a complete lattice.

To prove a result on existence define a norm on BM(I_1, R) by

$$\|x\|_1 = \sup_{t \in I_1} \left(|x(t)| + |x^1(t)| \right) \text{ for } x \in BM(I_1, R).$$

Then clearly BM(I_1,R) is a Banach space with the above norm. We shall now prove the existence of maximal and minimal solutions for the IVP (2.1). For this we need the following assumptions:
(f_1): f is bounded on I_1xRxR by k_3, $k_3>0$.
(f_2) : $f(t,\varphi(t),\varphi^1(t))$ is Lebesgue measurable for Lebesgue measurable functions φ,φ^1 on I_1, and
(f_3) : $f(t,x,x^1)$ is nondecreasing in both x and x^1 in R for *a.a.* $t\in I_1$.

Theorem 2.3 Assume that the hypotheses (f_1-f_3) hold. Then the IVP (2.1) has maximal and minimal solutions on I_1.

Proof . Define a sub set S of the Banach space BM(I_1,R) by

$$S=\{x\in BM(I_1,R):\|x\|_1\le k_3^*\} \quad (2.3)$$

where $k_3^*=\text{Max}\left(|a|+|b|A+k_3\frac{A^2}{2}+|b|+k_3A,\ k_3\right)$.

Clearly S is closed, convex and bounded sub set of the Banach space BM(I_1,R) and hence by definition $(S,\le)$ is a complete lattice. We define an operator
$T:S\to$ BM(I_1,R) by

$$Tx(t)=a+bt+\int_0^t(t-s)f(s,x(s),x^1(s))\,ds,\ t\in I_1. \quad (2.4)$$

Then

$$Tx^1(t)=b+\int_0^t f(s,x(s),x^1(s))ds,\ t\in I_1$$

Obviously (Tx) and (Tx^1) are continuous on I_1 and hence measurable on I_1. We now show that T maps S into itself. Let $x\in S$ be an arbitrary point , then

$$|Tx(t)|\le|a|+|b|t+\int_0^t(t-s)\,|f(s,x(s),x^1(s))|\,ds$$

$$\le|a|+|b|t+k_3\frac{t^2}{2}$$

And

$$|Tx^1(t)|\le|b|+\int_0^t|f(s,x(s),x^1(s))|ds$$

$$\le|b|+k_3t$$

Therefore

$$\|Tx\|_1=\sup_{t\in I_1}\left(|Tx(t)|+|Tx^1(t)|\right)$$

$$\le\sup_{t\in I_1}\left(|a|+|b|t+k_3\frac{t^2}{2}+|b|+k_3t\right)$$

$$\le\left(|a|+|b|A+k_3\frac{A^2}{2}+|b|+k_3A\right)\le k_3^*$$

This shows that T maps S into itself. Let x,y be such that $x\le y$, then by (f_3) we get

$$Tx(t)=a+bt+\int_0^t(t-s)\ f(s,x(s),x^1(s))\,ds$$

$$\le a+bt+\int_0^t(t-s)\ f(s,y(s),y^1(s))\,ds=Ty(t)\ \ \forall t\in I_1$$

and

$$Tx^1(t)=b+\int_0^t f(s,x(s),x^1(s))ds$$

$$\le b+\int_0^t f(s,y(s),y^1(s))ds=Ty^1(t)\ ,t\in I_1.$$

This shows that T is isotone increasing on S.
In view of Theorem 2.2, it follows that the operator equation $Tx=x$ has solutions and that the set of all solutions is a complete lattice, implying that the set of all solutions of (2.1) is a complete lattice. Consequently the IVP (2.1) has maximal and minimal solutions in S.

Finally in view of the definition of the operator T, it follows that these extremal solutions are in C(I_1,R) $\subset$ AC(I_1,R). This completes the proof.

We shall now show that the maximal and minimal solutions of the IVP (2.1) serve as the bounds for the solutions of the differential inequalities related to the IVP (2.1).

Theorem 2.4 Assume that all the conditions of Theorem 2.3 are satisfied. Suppose that there exists a function u$\in$ S, where S is as defined in the proof of Theorem 2.3 satisfying

$$u^{11}\le f(t,u,u^1)\ \ a.a.\ t\in I_1 \text{ with } u(0)=a,\ u^1(0)=b \quad (2.5)$$

Then there exists a maximal solution x_M of the IVP (2.1) such that
$u(t)\le x_M(t),\ t\in I_1$.

Proof. Let p = Sup S. Clearly the element p exists, since S is a complete lattice. Consider the lattice interval [u, p] in S where u is a solution of (2.5) .We notice that [u, p] is obviously a complete lattice.

It can be shown as in the proof of Theorem 2.3 that T: [u, p] $\to S$ is isotone increasing on [u,p] . We show that T maps [u, p] into itself. For this it suffices to show that $u\le Tx$ for any x$\in S$ with $u\le x$. Now from the inequality (2.5), it follows that

$$u^1(t)\le b+\int_0^t f(s,u(s),u^1(s))ds$$

$$\le b+\int_0^t f(s,x(s),x^1(s))ds=Tx^1(t)\ ,\ t\in I_1 \text{ And}$$

$$u(t)\le a+bt+\int_0^t(t-s)f(s,u(s),u^1(s))\,ds$$

$$\le a+bt+\int_0^t(t-s)f(s,x(s),x^1(s))\,ds=Tx(t) \text{ for all } t\in I_1.$$

This shows that T maps [u, p] into itself. Applying the Theorem 2.2, we conclude that there is a maximal solution x_M of the integral equation (2.2) and consequently of the IVP (2.1) in [u, p].

Therefore
$u(t) \le x_M(t)$ for $t \in I_1$.
This completes the proof.

Theorem 2.5 Suppose that all the conditions of Theorem 2.3 hold, and assume that there is a function $v \in S$, where S is as defined in the proof of Theorem 2.3, such that
$v^{11} \ge f(t, v, v^1)$ *a.a.* $t \in I_1$ with $v(0) = a$, $v^1(0) = b$.

Then there is a minimal solution x_m of the IVP (2.1) such that $x_m(t) \le v(t)$, for $t \in I_1$.

The proof is similar to that of Theorem 2.4 and we omit the details We shall now prove the uniqueness of solutions of the IVP (2.1).

Theorem 2.6 In addition to the hypothesis of Theorem 2.3, if the function $f(t, x, x^1)$ on I_1 x R x R satisfies the condition that

$$\left| f(t,x,x^1) - f(t,y,y^1) \right| \le M \;\; Min\left(\frac{|x-y|}{A+|x-y|}, \frac{|x^1-y^1|}{A+|x^1-y^1|} \right) \quad (2.6)$$

for some $M>0$. Then the IVP (2.1) has unique solution defined on I_1 .

Proof . Let BM(I_1, R) denote the space of all bounded and measurable functions defined on I_1 . Define a norm on BM(I_1, R) by

$$\|x\|_2 = \sup_{t \in I_1} e^{-Lt} |x(t)| \text{ , for } x \in \text{ BM } (I_1, \text{R})$$

where L is a fixed positive constant such that $\frac{M}{L^2} < 1$ .Notice that BM(I_1, R) is a Banach space with the norm $\|.\|_2$. Define T: BM(I_1,R) $\longrightarrow$ BM(I_1,R) by (2.4).

Let $x_m(t)$, $x_M(t)$ be minimal and maximal solutions of the IVP (2.1) respectively. Then we have
$(T\,x_M)(t)$ - $(T\,x_m)(t)$ =

$$\int_0^t (t-s)\left[f(s, x_M(s), x_M{}^1(s)) - f(s, x_m(s), x_m{}^1(s)) \right] ds .$$

Using (2.6) we see that

$$\left| (T\,x_M)(t) - (T\,x_m)(t) \right|$$

$$\le M \int_0^t (t-s) \left(\frac{|x_M(s) - x_m(s)|}{A + |x_M(s) - x_m(s)|} \right) ds$$

That is

$$\sup_{t \in I_1} \left| (T\,x_M)(t) - (T_3 x_m)(t) \right| \le M \;\; \|x_M - x_m\|_2 \int_0^t (t-s) e^{Ls} ds$$

$$\le M \;\; \|x_M - x_m\|_2 \;\; \frac{e^{Lt}}{L^2} .$$

This implies that

$$\|T\,x_M - T\,x_m\|_2 \le \frac{M}{L^2} \;\; \|x_M - x_m\|_2 .$$

Since $\frac{M}{L^2} < 1$, the mapping T is a contrition .Applying the contraction mapping principle, we conclude that there exists a unique fixed point $x \in$ BM(I_1,R) such that

$(Tx)(t)$ =

$$x(t) = a + bt + \int_0^t (t-s) f(s, x(s), x^1(s)) ds \;\; \forall \;\; t \in I_1 \;\; .$$

This completes the proof.

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References

[1] B.C. Dhage and G.P. Patil “ On differential inequalities for discontinuous non-linear two point boundary value problems” Differential equations and dynamical systems, volume 6, Number 4, October 1998, pp. 405 – 412.

[2] B.C.Dhage , “On weak Differential Inequalities for nonlinear discontinuous boundary value problems and Applications’’. Differential equations and dynamical systems, volume 7, number 1, January 1999, pp.39-47.

[3] K. Narshmha Reddy and A.Sreenivas “On Integral Inequities in n – Independent Variables”, Bull .Cal .Math. Soc ., 101 (1) (2009), 63-70 .

[4] G. Birkhoff , “Lattice theory ” , Amer .Math. Soc. Coll.publ. Vol.25, NewYork, 1979.

[5] S.G. Deo, V.Lakshmikantham, V.Raghavendra, “Text book of Ordinary Differential Equations” second edition.

[6] B.C.Dhage “ Existence theory for nonlinear functional boundary value Problems’’. Electronic Journal of Qualitative theory of Differential Equations, 2004, No 1, 1-15.

[7] A. Tarski , “A lattice theoretical fixed point theorem and its application, pacific . J. Math., 5(1955), 285 – 310.