

An Analysis and Study of Iteration Procedures

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ABSTRACT

In computational mathematics, an iterative method is a scientific technique that utilizes an underlying speculation to produce a grouping of improving rough answers for a class of issues, where the n-th estimate is gotten from the past ones. A particular execution of an iterative method, including the end criteria, is a calculation of the iterative method. An iterative method is called joined if the relating grouping meets for given starting approximations. A scientifically thorough combination investigation of an iterative method is typically performed; notwithstanding, heuristic-based iterative methods are additionally normal.

This Research provides a survey of iteration procedures that have been used to obtain fixed points for maps satisfying a variety of contractive conditions.

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1. INTRODUCTION

The literature abounds with papers which establish fixed points for maps satisfying a variety of contractive conditions. In most cases the contractive definition is strong enough, not only to guarantee the existence of a unique fixed point, but also to obtain that fixed point by repeated iteration of the function. However, for certain kinds of maps, such as nonexpansive maps, repeated function iteration need not converge to a fixed point.

A none expansive map satisfies the condition $\|Tx - Ty\| \leq \|x - y\|$ for each pair of points x, y in the space. A simple example is the following. Define $T(x) = 1 - x$ for $0 \leq x \leq 1$. Then T is a none expansive self map of $[0,1]$ with a unique fixed point at $x = 1/2$, but, if one chooses as a starting point the value $x = a, a \neq 1/2$, then repeated iteration of T yields the sequence $\{1 - a, a, 1 - a, a, \dots\}$.

One example of such matrices is the Cesaro matrix, obtained by choosing $c_n = 1/(n + 1)$. Another is $c_n = 1$ for all $n \geq 0$, which corresponds to ordinary function iteration, commonly called Picard iteration.

In 1953 W.R. Mann defined the following iteration procedure. Let A be a lower triangular matrix with nonnegative entries and row sums 1. Define $x_{n+1} = T(v_n)$, where

$$v_n = \sum_{k=0}^n a_{nk} x_k.$$

Picard iteration of the function $S1/2 = (1+T)/2$ is equivalent to the Mann iteration scheme with

$$a_{nk} = \frac{1}{2^n} \binom{n}{k}$$

The most interesting cases of the Mann iterative process are obtained by choosing matrices A such that $a_{n+1, k} = (1 - a_{n+1, n+1})a_{nk}, k = 0, 1, \dots, n; n = 0, 1, 2, \dots$, and either $a_{nn} = 1$ for all $n > 0$. Thus, if one chooses any sequence $\{c_n\}$ satisfying (i) $C_0 = 1$, (ii) $0 \leq c_n < 1$ for $n > 0$, and (iii) $\sum c_n = \infty$ then the entries of A become $a_{nn} = c_n$

This matrix is the Euler matrix of order 1, and the transformation $S1/2$ has been investigated by Edelstein and Krasnoselskii [30]. Krasnoselskii showed that, if X is a uniformly convex Banach space, and T is a nonexpansive selfmap of X , then $S1/2$ converges to a fixed point of T . Edelstein showed that the condition of uniform convexity could be weakened to that of strict convexity. Picard iteration of the function $S\lambda = \lambda I + (1 - \lambda)T, 0 < \lambda < 1$, for any function T , homogeneous of degree 1, is equivalent to the

Mann iteration scheme with $a_{nk} = \binom{n}{k} \lambda^k (1 - \lambda)^{n-k}$

$$a_{nk} = c_k \prod_{j=k+1}^n (1 - c_j), k < n \dots \dots (1.1)$$

This matrix is the Euler matrix of order $(1 - \lambda)/\lambda$. The iteration of $S\lambda$ has been investigated by Browder and Petryshyn, Opial], and Schaefer. Mann showed that, if T is any continuous selfmap of a closed interval $[a, b]$ with at most one fixed point, then his iteration scheme, with $c_n = 1/(n + 1)$, converges to the fixed point of T . Franks and Marzecz extended this result to continuous functions possessing more than one fixed point in the interval. A matrix A is called a weighted mean matrix if A is a lower

and A is a regular matrix. (A regular matrix is a bounded linear operator on E^{∞} such that A is limit preserving for convergent sequences.) The above representation for A allows one to write the iteration scheme in the following form:
 $x_{n+1} = (1 - c_n)x_n + c_n T(x_n)$.

triangular matrix with nonzero entries $a_{nk} = p_k / P_n$ where $\{p_k\}$ is a nonnegative sequence with p_0 positive and

$$P_n = \sum_{k=0}^n p_k \rightarrow \infty$$

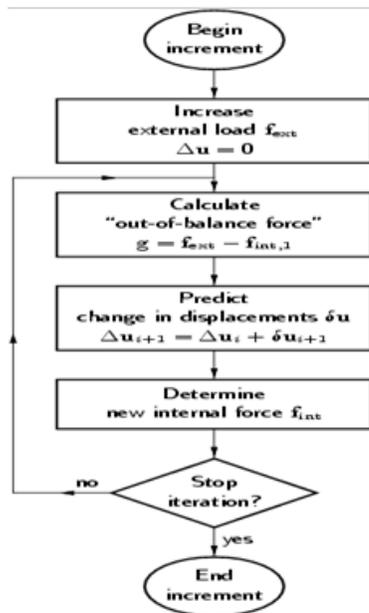
The author extended the above-mentioned result of Franks and Marzec to any continuous self map of an interval $[a, b]$, and any weighted mean matrix satisfying the condition

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n-1,k}| = 0.$$

In [42] the author also showed that the matrix defined by (1.1) is equivalent to a regular weighted mean matrix with weights

$$p_k = \frac{c_k p_0}{\prod_{j=1}^k (1 - c_j)}, k > 0.$$

Let E be a Banach space, C a closed convex subset of E , T a continuous selfmap of C . Mann showed that, if either of the sequences $\{x_n\}$ or $\{v_n\}$ converges, then so does the other, and to the same limit, which is a fixed point of T . Dotson extended this result to locally convex Hausdorff linear topological spaces E . Consequently, to use the Mann iterative process on nonexpansive maps, all one needs is to establish the convergence of either $\{x_n\}$ or $\{v_n\}$.



(Figure: Iteration Procedure)

For uniformly convex Banach spaces, the following was obtained independently by Browder Kirk and Gohde .

Let C be a closed, bounded, and convex subset of a uniformly convex Banach space, T a nonexpansive selfmap of C . Then T has a fixed point. Unfortunately the proofs of above theorem are not constructive. A number of papers have been written to obtain some kind of sequential convergence for nonexpansive maps. Most such theorems are valid only under some additional hypothesis, such as compactness, and converge only weakly. Halpern obtained two algorithms for obtaining fixed points for nonexpansive maps on Hilbert spaces. In his dissertation, Humphreys constructed an algorithm which can be applied to obtain fixed points for nonexpansive maps on uniformly convex Banach spaces. A nonexpansive mapping is said to be asymptotically regular if, for each point x in the space, $\lim(T^{n+1}x - T^nx) = 0$. In 1966 Browder and Petryshyn established the following result.

2. Theorems:

2.1 THEOREM 1: Let X be a Banach space, T a nonexpansive asymptotically regular selfmap of X . Suppose that T has a fixed point, and that $I - T$ maps bounded closed subsets of X into closed subsets of X . Then, for each $x_0 \in X$, $\{T^n x_0\}$ converges to a fixed point of T in X .

In 1972 Groetsch established the following theorem, which removes the hypothesis that T be asymptotically regular.

2.2 THEOREM 2. Suppose T is a nonexpansive selfmap of a closed convex subset E of X which has at least one fixed point. If $I - T$ maps bounded closed subsets of E into closed subsets of E , then the Mann iterative procedure, with $\{c_n\}$ satisfying conditions (i), (ii), and (iv) $\sum c_n(1 - c_n) = \infty$, converges strongly to a fixed point of T . Ishikawa established the following theorem.

2.3 THEOREM 3. Let D be a closed subset of a Banach space X and let T be a no expansive map from D into a compact subset of X . Then T has a fixed point in D and the Mann iterative process with $\{c_n\}$ satisfying conditions (i) - (iii), and $(0 < c_n < b < 1$ for all n , converges to a fixed point of T .

For spaces of dimension higher than one, continuity is not adequate to guarantee convergence to a fixed point, either by repeated function iteration, or by some other iteration procedure. Therefore it is necessary to impose some kind of growth condition on the map. If the contractive condition is strong enough, then the map will have a unique fixed point, which can be obtained by repeated iteration of the function. If the contractive condition is slightly weaker, then some other iteration scheme is required. Even if the fixed point can be obtained by function iteration, it is not without interest to determine if other iteration procedures converge to the fixed point. A generalization of a nonexpansive map with at least one fixed point that of a quasi-nonexpansive map. A function T is a quasi-nonexpansive map if it has at least one fixed point, and, for each fixed point p , $\|Tx - p\| < \|x - p\|$. The following is due to Dotson.

2.4 THEOREM 4. Let E be a strictly convex Banach space, C a closed convex subset of E , T a continuous quasi-nonexpansive selfmap of C such that $T(C) \subset K \subset C$, where K is compact. Let $x_0 \in C$ and consider a Mann iteration process such that $\{c_n\}$ clusters at some point in $(0,1)$. Then the sequences $\{x_n\}$, $\{v_n\}$ converge strongly to a fixed point of T . A contractive definition which is included in the class of quasi-contractive maps is the following, due to Zamfirescu . A map satisfies condition Z if, for each pair of points x,y in the space, at least one of the following is true:

- (i) $\|Tx - Ty\| < \alpha \|x - y\|$, (ii) $\|Tx - Ty\| < \beta [\|x - Tx\| + \|y - Ty\|]$, or
- (iii) $\|Tx - Ty\| < \gamma [\|x - Ty\| + \|y - Tx\|]$, α, β, γ where are real nonnegative constants satisfying $\alpha < 1, \beta, \gamma < 1/2$ As shown in [52], T has a unique fixed point, which can be obtained by repeated iteration of the function. The following result appears in.

2.5 THEOREM 5. Let X be a uniformly convex Banach space, E a closed convex subset of X , T a self map of E satisfying condition Z . Then the Mann iterative process with $\{c_n\}$ satisfying conditions (i), (ii), and (iv) converges to the fixed point of T . A generalization of definition Z was made by Ciric [11]. A map satisfies condition C if there exists a constant k satisfying $0 \leq k < 1$ such that, for each pair of points x, y in the space,

$$\|Tx - Ty\| < k \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}.$$

In [42] the author proved the following for Hilbert spaces.

2.6 THEOREM 6. Let H be a Hilbert space, T a selfmap of H satisfying condition C . Then the Mann iterative process, with $\{c_n\}$ satisfying conditions (i)-(iii) and $\limsup c_n < 1 - k_2$ converges to the fixed point of T . Chidume [10] has extended the above result to l_p spaces, $p \geq 2$, under the conditions $k_2(p - 1) < 1$ and $\limsup c_n < (p - 1)_1 - k_2$. As noted earlier, if T is continuous, then, if the Mann iterative process converges, it must converge to a fixed point of T . If T is not continuous, there is no guarantee that, even if the Mann process converges, it will converge to a fixed point of T . Consider, for example, the map T defined by $T0 = T1 = 0, Tx = 1, 0 < x < 1$. Then T is a selfmap of $[0,1]$, with a fixed point at $x = 0$. However, the Mann iteration scheme, with $c_n = 1/(n + 1), 0 < x_0 < 1$, converges to 1, which is not a fixed point of T . A map T is said to be strictly-pseudo contractive if there exists a constant $k, 0 < k < 1$ such that, for all points x, y in the space,

$$\|Tx - Ty\| \leq \|x - y\| + k\|(I - T)x - (I - T)y\|.$$

We shall call denote the class of all such maps by P_2 . Clearly P_2 mappings contain the nonexpansive mappings, but the classes P_2, C , and quasi-nonexpansive mappings are independent.

3. STABILITY.

We shall now discuss the question of stability of iteration processes, adopting the definition of stability that appears in [1]. Let X be a Banach space, T a selfmap of X , and assume that $x_{n+1} = f(T, x_n)$ defines some iteration procedure involving T . For example, $f(T, x_n) = Tx_n$. Suppose that $\{x_n\}$ converges to a fixed point p of T . Let $\{y_n\}$ be an arbitrary sequence in X and define $e_n = \|y_{n+1} - f(T, y_n)\|$

for $n = 0, 1, 2, \dots$. If $\lim e_n = 0$ implies that $\lim y_n = p$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T -stable. The first result on T -stable mappings was proved by Ostrowski for the Banach contraction principle. In the authors show that function iteration is stable for a variety of contractive definitions. Their best result for function iteration is the following.

3.1 THEOREM. Let X be a complete metric space, T a selfmap of X satisfying the contractive condition of Zamfirescu. Let p be the fixed point of T . Let $x_0 \in X$, set $x_{n+1} = Tx_n, n > 0$. Let $\{y_n\}$ be a sequence in X , and set $e_n = d(y_{n+1}, Ty_n)$ for $n = 0, 1, 2, \dots$. Then

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^n 2\delta^{n+1-k} d(x_k, x_{k+1}) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^n \delta^{n-k} e_k$$

$$\delta = \max \left\{ \alpha, \frac{\beta}{1 - \beta}, \frac{\gamma}{1 - \gamma} \right\}$$

$$\lim_{n \rightarrow \infty} y_n = p$$

For the Mann iteration procedure their best result is the following.

3.2 THEOREM. Let $(X, \|\cdot\|)$ be a normed linear space, T a selfmap of X satisfying the Zamfirescu condition. Let $x_0 \in X$, and suppose that there exists a fixed point p and $x_n \rightarrow p$, where $\{x_n\}$ denotes the Mann iterative procedures with the $\{c_n\}$ satisfying (i), (ii), and $0 \leq a \leq c_n \leq b < 1$. Suppose $\{y_n\}$ is a sequence in X and $e_n = \|y_{n+1} - [(1 - c_n)y_n + c_nTy_n]\|$ for $n = 0, 1, 2, \dots$. Then

$$\|p - y_{n+1}\| \leq \|p - x_{n+1}\| + (1 - a + a\delta)^{n+1} \|x_0 - y_0\| +$$

$$\sum_{i=0}^n 2\delta b(1 - a + a\delta)^{n+1} \|x_i - Tx_i\| + \sum_{i=0}^n (1 - a + a\delta)^{n-i} e_i, \quad n = 0, 1, 2, \dots,$$

where

$$\delta = \max \left\{ \alpha, \frac{\beta}{1 - \beta}, \frac{\gamma}{1 - \gamma} \right\}$$

$$\lim_{n \rightarrow \infty} y_n = p \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} e_n = 0.$$

For the iteration method of Kirk, they have the following result.

Conclusion:

Iteration is the redundancy of a procedure so as to create a (perhaps unbounded) succession of results. The grouping will approach some end point or end esteem. Every redundancy of the procedure is a solitary iteration, and the result of every iteration is then the beginning stage of the following iteration. In mathematics and software engineering, iteration (alongside the related system of recursion) is a standard component of calculations. In algorithmic circumstances, recursion and iteration can be utilized to a similar impact. The essential distinction is that recursion can be utilized as an answer without earlier learning about how often the activity should refresh, while an effective iteration necessitates that premonition.

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