Some Characteristics of Weibull Distribution and its Contribution to Wind Energy Analysis

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ABSTRACT

Weibull distribution is a proper distribution to use in modelling the life time data for more than four decades. The Weibull distribution has been found to be a suitable fit for the empirical distribution of the wind speed measurement samples. In brief this paper consist of presentation of general review of important properties and characters in Weibull distribution. Also we discuss the application of the two-parameter Weibull distribution to wind speed measurements. The estimation of unknown parameters in Weibull distribution along with some properties of moment method are also discussed.

Keywords: Weibull distribution, Wind speed, Estimators, Moment method

INTRODUCTION

Towards the end of 20th and beginning of the 21st centuries, interest has risen in new and renewable energy sources especially wind energy for electricity generation. The scientists and researchers attempted to accelerate solutions for wind energy generation design parameters. Our life is directly related to energy and its consumption, and the issues of energy research are extremely important and highly sensitive. In a short time, wind energy is welcomed by society, industry and politics as a clean, practical, economical and environmentally friendly alternative. Wind energy has recently been applied in various industries, and it started to compete with other energy resources. Wind energy history, wind-power meteorology, the energy climate relations, wind-turbine technology, wind economy, wind–hybrid applications and the current status of installed wind energy capacity all over the world reviewed critically with further enhancements and new research trend direction suggestions.

Wind energy can be utilized for a variety of functions ranging from windmills to pumping water and sailing boats. With increasing significance of environmental problems, clean energy generation becomes essential in every aspect of energy consumption. Wind energy is very clean but not persistent for long periods of time. In potential wind energy generation studies fossil fuels must be supplemented by wind energy. There are many scientific studies in wind energy domain, which have treated the problem with various approaches [1-3]. General trends towards wind and other renewable energy resources increased after the energy crises of the 20th century [4].

As a meteorological variable, wind describes fuel of wind energy. In energy production, wind takes the same role as water, and wind variables should be analyzed. Wind speed deviation and changeability depend on time and area. This situation requires a new tendency for wind-speed modeling and search for the atmospheric boundary layer modeling as a special consideration in wind-power research. There are many papers concerning these subjects. Wind energy and speed change with time and are not continual at the same area during the whole year. Wind speed is a regionalized variable measured at a set of irregular sites. Wind energy investigations mostly rely upon arithmetic average of the wind speed. However, many authors based the wind energy estimates on elaborated wind-speed statistics, including the standard deviation, skewness and kurtosis coefficient. Some researches advocate the use of two-parameter Weibull

distribution in wind velocity applications. Their suggestions are taken as granted in many parts of the world for wind energy calculations.

The aim of this paper is to review the characteristics and properties of the Weibull distribution. Then the interpretation of the parameters using moment method and the contribution of Weibull distribution to wind speed evaluation are discussed.

Weibull Distribution:
In the early 1920s, there were three groups of scientists working on the derivation of the distribution independently with different purposes. Waloddi Weibull was one of them [5,6]. The distribution bears his name because he promoted this distribution both internationally and interdisciplinary. His discoveries lead the distribution to be productive in engineering practice, statistical modeling and probability theory. The Weibull distribution is widely used in reliability [7].Thousands of papers have been written on this distribution and it is still drawing broad attention. It is of importance to statisticians because of its ability to fit to data from various areas, ranging from life data to observations made in economics, biology or materials reliability theory. The Weibull distribution is the most widely used life time distribution. It is used as a model for life times of many manufactured items such as vacuum tubes, ball bearings and electrical insulation. It is also used in biomedical applications, for example as the distribution for life times to occurrence (diagnosis) of tumors in human populations or in laboratory animals. Further the Weibull distribution is the suitable model for extreme value data such as the strength of certain materials. Recently empirical studies have shown that the Weibulll distribution is superior to the classical stable distributions inclusive the normal distributions, for fitting empirical economic data.

Probability density function:
The Weibull distribution gives a good match with the experimental data. This is characterized by the shape parameter k and scale parameter c(m/s). The Weibull distribution is often used in the field of life data analysis due to its ability to fit the Gamma distribution and normal distribution and interpolate a range of shapes in between them.

The probability density function is given by

$$f(v) = \left(\frac{k}{c}\right)\left(\frac{v}{c}\right)^{k-1} exp\left(-\left(\frac{v}{c}\right)^k\right) \qquad (1)$$

and the cumulative distribution function is

$$F(v) = 1 - exp\left(-\left(\frac{v}{c}\right)^k\right) \qquad (2)$$

where v is the wind speed, k is the shape parameter and c is the scale parameter.

Interpretation of Weibull parameters:
The aim of this section is to review the parameters namely the shape parameter, scale parameter and hazard rate of the distribution. Also the interpretation of the parameters and their physical meaning will be introduced.

Shape parameter k:
Weibull is a versatile distribution that can take on the characteristics of other types of distributions based on the value of the shape parameter. The value of k is equal to the slope of the line in a probability plot on Weibull probability

paper. The value of shape parameter has remarkable effect on the behavior of Weibull distribution [8,9]. The following plot shows the effect of different values of the shape parameter k.

TABLE 1 WEIBULL DISTRIBUTION SHAPE PARAMETER K PROPERTIES

Shape Parameter	Probability Density Function
0<k<1	Exponentially decay from infinity
k=1	Exponentially decay from /mean
1<k<2	Rises to peak and then decreases
k=2	Rayleigh distribution
$3 \le k \le 4$	Has normal bell shape appearance
k>10	Has shape very similar to type 1 extreme value distribution

FIGURE1 WEIBULL DISTRIBUTION PROFILE AT VARIOUS K
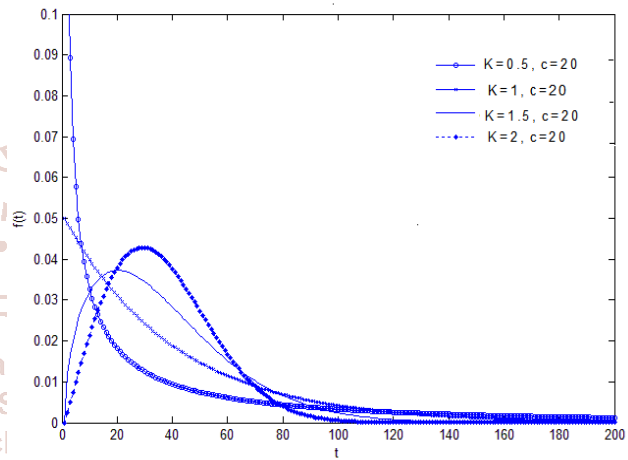
Scale parameter c
The value of scale parameter c has a different effect on the Weibull distribution. It is related to the location of the central portion along the abscissa scale. s From Figure 2 we can say that
➢ If c is increased, while k is constant, the Weibull distribution gets stretched out to the right and its height lowers.
➢ If c is decreased, while k is constant, the Weibull distribution gets pushed in towards the left, and its height increases.
➢ This is because the area under the density must be unity.

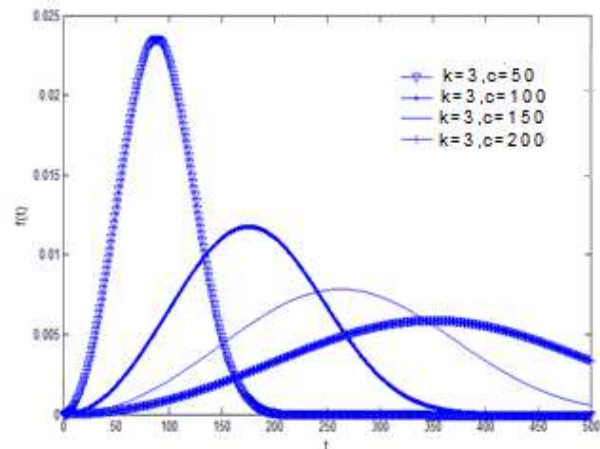FIGURE 2 WEIBULL DISTRIBUTION PROFILE AT VARIOUS C

TABLE 1 CHARACTERISTIC OF WIND SPEED

Annual mean wind speed	At 10 m Height Indicated value of wind resource
< 4.5 m/s	Poor
4.5 – 5.4 m/s	average
5.4 – 6.7 m/s	Good
> 6.7 m/s	Exceptional

Concept of Hazard rate

The parameter of interest here is the hazard rate, h(v). This is the conditional probability that an Equipment will fail in a given interval of unit time given that it has survived until that interval of time. It is, therefore, the instantaneous failure rate and can in general be thought of as a measure of the probability of failure, where this probability varies with the time the item has been in service. The 3 failure regimes are defined in terms of hazard rate and not, as is a common misconception, in terms of failure rate. [10]

Definition 1:

One of the most important quantities characterizing life phenomenon in life testing analysis is the hazard rate function defined by $h(v) = \frac{f(v)}{1-F(v)}$

The hazard rate for the Weibull distribution is expressed by $h(v) = \frac{k}{c}\left(\frac{v}{c}\right)^{k-1}$

The cumulative hazard function is given by

$$H(v) = \int h(v) = \left(\frac{v}{c}\right)^k$$

Lemma1:

The hazard rate function of a Weibull distribution has the following properties.
1. If k=c=1 then the hazard rate is unit.
2. Only if c=1 then the hazard rate is increasing for k>1 and decreasing for k<1.
3. Only if k=1 then the hazard rate is other than unity.

Proof:

1. It is clear that k=c=1 then h (v)=1.
2. If c=1 then $h(v)=kv^{k-1}$ which is increasing for k>1 and decreasing for k<1.
3. It is clear that if k=1 then h(v)=1/c. For different values of c it will be a constant other than unit.

Some of the properties of Weibull Distribution:

In this section we have discussed some of the properties of Weibull distribution and its associated theorems. In my previous work I have used two mixture probability distributions namely Weibull-Gamma and Weibull-Normal. So here we mentioned the mean and variance of those properties also.

➤ If X has a Weibull distribution then its mean

$$E(X) = c \, \Gamma\left(1 + \frac{1}{k}\right)$$

$$var(X) = c^2\left[\Gamma\left(1 + \frac{1}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right)\right]$$

r^{th} moment about the origin is $E(X^r) = \lambda^r \Gamma\left(1 + \frac{r}{c}\right)$

➤ The skewness is given by

$$\gamma_1 = \frac{\Gamma\left(1 + \frac{3}{k}\right)c^3 - 3\mu\sigma^2 - \mu^3}{\sigma^3}$$

where the mean is denoted by μ and the standard deviation is denoted by σ.

➤ The kurtosis is given by

$$\gamma_2 = \frac{-6\Gamma_1^4 + 12\Gamma_1^2\Gamma_2 - 3\Gamma_1\Gamma_3 + \Gamma_4}{\left[\Gamma_2 - \Gamma_1^2\right]^2}$$

where $\Gamma_i = \Gamma\left(1 + \frac{i}{k}\right)$

➤ A main reason for its popularity is that it has a great variety of shapes this makes it extremly flexible in fitting data, and it has been found to provide a good description of many kinds of data.

➤ As $\iota \to \alpha$ the hazard rate of the distribution increases to infinity if $k < 1$ and is constanat for $k = 1$. Among all usual life time distributions no other distribution has this property. This property makes the Weibull distribution suitable in applications where a decreasing hazard tendency distribution or an increasing hazard tendency distribution is required.

➤ It contains the exponential distribution as a special case where $k = 1$, further it has the following strong relation to the exponential distribution. If X has a Weibull distribution then X^k is exponentially distributed. So by appropriate changing of the scale we get the exponential distribution from the Weibull vice versa.

➤ The Weibull distribution is one of the three possible types of minimum stable distributions. Let $X_1, X_2 \ldots \ldots$ be identically independenty distributed random variables with common distribution function F. If for each n there exists constants $a_n > 0$ and b_n such that

$$a_n \underset{n}{\min}\{X_1, X_2 \ldots \ldots, X_n\} + b_n = X \quad \text{then the}$$

distribution of X is called minimum stable. It can be shown that there exist three types of such distributions.

➤ If X_i have a Weibull distribution when shape parameter k, it is easily shown that $n^{\frac{1}{k}}\underset{n}{\min}\{X_1, X_2 \ldots \ldots, X_n\} = X$. This stability property is an expression of a deeper limiting property.

➤ If F(0)=0 and F(t) behaves as ct^k for some possitive c and k as $t \to 0$ from above then $n^{\frac{1}{k}}\underset{n}{\min}\{X_1, X_2 \ldots \ldots, X_n\}$ converges in distribution to the Weibull distribution with shape parameter k. This characteristics property makes the Weibull distribution a natural model for extreme value data

Theorem 1:

The inverse Weibull cumulative function can be expressed in closed form.

Proof:

The probability density function and cumulative distribution of Weibull are (1) and (2) respectively.

Equating (2) to u where 0<u<1, we get $F^{-1}(v) = c[-\ln(1-u)]^{1/k}$.
Which shows that the inverse Weibull cumulative function can be expressed in closed form.

Theorem 2:
The Weibull Distribution has the scaling property.

Proof:
Therefore by the transformation technique the probability density function of U is

$$f_U(u) = f_v[g^{-1}(u)]\left|\frac{dv}{du}\right| = \left(\frac{k}{tc}\right)\left(\frac{u}{tc}\right)^{k-1} exp\left(-\left(\frac{u}{tc}\right)^k\right),$$

which shows the scaling property of Weibull distribution.

Mixture Distributions:
A mixture distribution is a weighted average of probability distribution of positive weights that sum to one. The distributions thus mixed are called the components of the mixture. The weights themselves comprise a probability distribution called the mixing distribution.

Definition 2:
A random variable X is said to have a weibull mixtured distribution if its probability density function is defined as

$$f(x,k,c,\alpha) = \int_0^\infty kc\, r^{c-1}e^{-ar^c} g(x,\alpha)dr \qquad (3)$$

where $g(x,\alpha)$ is a probability density function. The name of weibull mixture distribution comes from the fact that the distribution (3) is the weighted average of $g(x,\alpha)$ with weights equal to the ordinates of Weibull distribution.

Definition 3:
If X follows a Weibull Mixture of Normal distribution with parameters k and c then the density function is given by

$$f(x,k,c) = \int_0^\infty kc\, r^{c-1}e^{-ar^c} \frac{e^{-x^2/2x^{2r}}}{2^{r+\frac{1}{2}}\sqrt{(r+\frac{1}{2})}} dr , -\infty < x < \infty$$ with parameters k and c such that $\int_{-\infty}^\infty f(x,k,c)dx = 1$.

Proposition 1:
If X has a Weibull mixtures of Normal Distribution with parameters k and c then its mean and variance are 0 and $1 + 2k^{\frac{1}{c}}\sqrt{(1+\frac{1}{c})}$ respectively.

Definition 4:
A random variable X having the density function

$$f(x,k,c,\alpha,\beta) = \int_0^\infty kc\, r^{c-1}e^{-ar^c} \frac{\beta^{x+r}e^{-\beta x}x^{\alpha+r-1}}{\sqrt{(\alpha+r)}} dr, \quad x > 0$$

is defined as Weibull Mixture Gamma Distribution with parameters k, c, α and β where as $\int_0^\infty f(x,k,c,\alpha,\beta)\, dx = 1$

Proposition 2:
If X denotes a Weibull Mixture of Gamma variate with parameters k, c, α and β then its mean and variance are

$$\frac{1}{\beta}\left[\alpha + k^{\frac{1}{c}}\sqrt{1+\frac{1}{b}}\right]$$

And

$$\frac{1}{\beta^2}[\alpha + k^{\frac{1}{c}}\sqrt{(1+\frac{1}{b})} + k^{2/c}\sqrt{(1+\frac{2}{c})} - \{k^{-1/c}\sqrt{(1+\frac{1}{c})}\}^2$$

respectively.

Estimation of the parameters using the method of moments:
This section discusses the statistical fitting of wind speed measurement data to the Two-parameter Weibull distribution, which has been recognized as a suitable fit for wind speed distribution .To be more precise, what is meant of "wind speed" here is the periodic (e.g. per minute) measurement of wind speed at a particular location, height and in a certain direction (e.g. southward). Clearly, wind has

Let t be a positive real constant. The transformation $U = g(V) = tV$ is a 1-1 transformation from $U = \{u/u > 0\}$ and $V = \{v/v > 0\}$ with inverse $= g^{-1}(U) = \frac{u}{t}$.

Also the Jacobian is given by $\frac{dv}{du} = \frac{1}{t}$.

a velocity at any given time, i.e. a speed as well as a direction. Wind speed measurements, thus, correspond to the magnitude of wind velocity in the measurement direction, which is argued to closely resemble a Weibull distribution [11, 12].

Weibull distribution (Weibull, 1951) has many applications in engineering and plays an important role in reliability and maintainability analysis. The Weibull distribution is one of

the extreme-value distributions which is applied also in optimality testing of Markov type optimization algorithms (Haan (1981), Zilinskas & Zhigljavsky (1991), Bartkute & Sakalauskas (2004)). Because of useful applications, its parameters need to be evaluated precisely, and efficiently. The parameter estimation will be primarily explained in the following chapters. Apart from the simplicity and ease with which the parameter estimates and the estimated distribution function can be constructed, the wide applicability of Weibull distribution depends on several attractive features which it possesses and in the following we mention most important moment method. Normally there are two different approaches for obtaining the point estimator for parameter is known. Namely classical method and decision theoretic approach. Now we outline some of the most important methods for obtaining estimators. Most commonly used methods under classical estimation are as follows

Definition 6:
Any function of the random sample $X_1, X_2, \ldots., X_n$ that are being observed say T $(X_1, X_2, \ldots., X_n)$ is called a **Statistic**.

Definition 5:
If a statistic is used to estimate an unknown parameter θ of a distribution, then it is called an **Estimator** and a particular value of the estimator say $T_n(X_1, X_2, \ldots., X_n)$ is called an estimate of θ. The process of estimating an unknown parameter is known as estimation.

Some characteristics of estimators:
Various statistical properties of estimators can be used to decide which estimator is most appropriate in a given situation.

Biased and Unbiased estimator:
➤ An estimator $\bar{\theta}$ is said to be unbiased for θ if $E(\bar{\theta}) = \theta$.
➤ If $E(\bar{\theta}) \neq \theta$ then we say that $\bar{\theta}$ is biased.
➤ In general the bias of an estimator is $B(\bar{\theta}) \equiv E(\bar{\theta}) = \theta$.
➤ If $B(\bar{\theta}) > 0$ then $\bar{\theta}$ over estimates θ.
➤ If $B(\bar{\theta}) < 0$ then $\bar{\theta}$ under estimates θ.
➤ If $\bar{\theta}$ is unbiased then of course $B(\bar{\theta}) = 0$.

Good Estimators:
In general a good estimator θ has the following properties.
➤ $\bar{\theta}$ is unbiased for θ
➤ $\bar{\theta}$ has small variance.

Unbiasedness:
A statistic T is an unbiased estimator of the parameter θ iff
$$E(T) = \theta$$

Example: Let X_1, X_2, X_3 be a random sample of size three from a Normal population of $N(\mu, \sigma^2)$. The statistic $T = \frac{1}{4}(X_1 + 2X_2 + X_3)$ is an unbiased estimate of μ.
Since $E(T) = E[\frac{1}{4}(X_1 + 2X_2 + X_3)]$
$= \frac{1}{4}(\mu + 2\mu + \mu)$
$= \mu$

Method of moment's fitting:
It can be shown that the theoretical moments of a distribution are reasonably approximated by their observational counterparts when the number of sample values is sufficiently large. The method of moments utilizes this fact to estimate the parameters of a distribution. Theoretical moments, which are functions of the unknown parameters, are equated to their corresponding sample moments. Estimates of the parameters are obtained by setting up as many equations as there are parameters, and solving these equations in terms of the sample moments. The goal of Moment method is to estimate the parameters that yields theoretical value of the distribution . The first two moments of the Weibull Distribution are given as

$$m_1 = c\ \Gamma\left(1 + \frac{1}{k}\right) \text{ and } m_2 = c^2 \Gamma\left(1 + \frac{2}{k}\right).$$

Let $H(k) = \frac{m_2}{m_1^2} = \frac{\Gamma\left(1 + \frac{2}{k}\right)}{\Gamma\left(1 + \frac{1}{k}\right)^2}$ \hfill (4)

Also $c = \frac{m_1}{\Gamma\left(1 + \frac{1}{k}\right)}$

Preposition 3:
The function H(k) is strictly monotone decreasing, has a unique root in $k \in (0, \infty)$ that converges to a negative value as $k \to \infty$.

Proof:
The derivative of H(k) is given by
$$\frac{dH(k)}{dx} = 2\,H(k)\left[\Psi\left(1 + \frac{1}{k}\right) - \Psi\left(1 + \frac{2}{k}\right)\right]$$
where $\Psi(y) = \Gamma'(y)/\Gamma(y)$. From calculus we know that if the derivative of a continuous function is continuous and negative, the function itself is strictly monotone decreasing. Since H(k) is unique we conclude that $\frac{dH(k)}{dx} < 0$ and thus the solution k is unique.

Theorem 3:
Let $\{v_i, i=1,2,\ldots n\}$ be a finite set of wind speed observations where $v_i \geq 0\ \forall i \in \{1,2,\ldots.n\}$
and $\forall j \in \{1,2,\ldots.n\}$ Such that $v_j - v_i > 0$. Then for H(k)>1 and there exists a unique pair of $k \in (0, \infty)\ and\ c \in (0, \infty)$ that gives $H(k) = \frac{m_2}{m_1^2}$ and $c = \frac{m_1}{\Gamma\left(1 + \frac{1}{k}\right)}$.

Proof:
Always $m_2 - m_1^2 \geq 0$ and $\frac{m_2}{m_1^2} > 1$.

Gamma function is always continuous on $(0, \infty]$. Hence (4) continuous.

Now $\underset{k \to \infty}{lt}\ H(k) = \underset{k \to \infty}{lt}\ \frac{\Gamma\left(1 + \frac{2}{k}\right)}{\Gamma\left(1 + \frac{1}{k}\right)} = 1.$

Also $\underset{k \to 0^+}{lt}\ H(k) = \underset{k \to 0^+}{lt}\ \frac{\Gamma\left(1 + \frac{2}{k}\right)}{\Gamma\left(1 + \frac{1}{k}\right)} \to +\infty$

Using intermediate theorem for the above results we can say that there exists a solution $k \in (0, \infty)$ to $H(k) = \frac{m_2}{m_1^2}$. As well as by proposition the solution k is unique. Therefore for a unique k then $c = \frac{m_1}{\Gamma\left(1 + \frac{1}{k}\right)}$. Concluding the proof[14].

Illustration of moment method using example:
Here we have presented four numerical methods namely Graphical method(GM), Emperical Method(EM), Moment method(MM) and Energy pattern Factor method(EPFM) for estimating Weibull parameters using wind speed data collected at Jogimatti located in Chitradurga district, Karnataka, India over a period of 20 years with mast height of 20m at Latitude N at 14° 09"49', Longitude E at 76° 23"56'and its performance is compared to others by using RMSE, χ^2 and R^2 statistical tests .
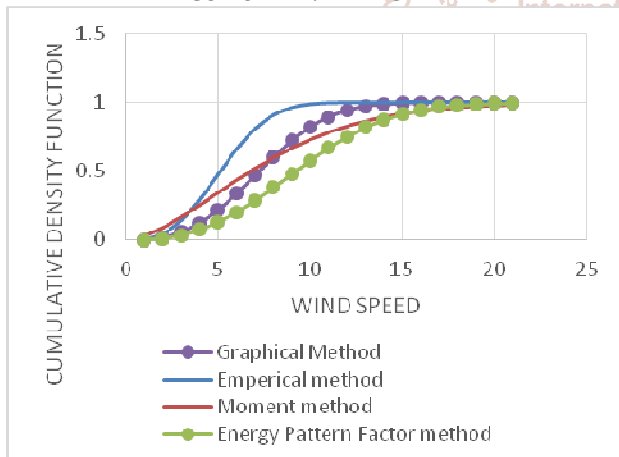
TABLE II-The Weibull parameters k and c

Weibull parameters	GM	EM	MM	EPFM
k	2.6260	2.5987	2.5801	2.5010
c	9.1901	9.2130	9.2142	9.2030

TABLE: III-Statistical Tests

TESTS	GM	EM	MM	EPFM
RMSE	0.0319	0.0315	0.0314	0.0306
χ^2	0.0011	0.0010	0.0010	0.0009
R^2	0.9585	0.9595	0.9599	0.9619

FIGURE 3 COMPARISON BEWEEN THE ESTIMATED CUMULATIVE DENSITY

SUMMARY
In this paper some mathematical properties of Weibull distribution are addressed. Also moment method or estimating the parameters of the Weibull wind speed distributions are described. For a sample wind speed data, the accurate parameter estimates obtained by moment method are presented. This study presents mathematical proofs of such estimates which is required for accurate wind power production simulation.

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