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A Hyperbolic PDE-ODE System with Delay-Robust Stabilization

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ABSTRACT
This paper is concerned with a new development of a delay-robust stabilizing feedback control law for linear ordinary differential equation coupled with two linear first order hyperbolic equations in the actuation path. A second change of variables that reduces the stabilization problem of the PDE-ODE system to that of a time-delay system for which a forecaster can be constructed. Hence, by choosing the pole placement on the ODE when constructing the forecaster, enabling a trade-off between convergence rate and delay-robustness. The proposed feedback law is finally proved to be robust to small delays in the actuation.

***Keyword:** Hyperbolic partial differential equation, Time-delay systems, Stabilization.*

INTRODUCTION

In this paper we develop a linear feedback control law that achieves delay-robust stabilization of a system of two hetero directional first-order hyperbolic Partial Differential Equations (PDEs) coupled through the boundary to an Ordinary Differential Equation (ODE). It has been observed, that for many feedback systems, the introduction of arbitrarily small time-delays in the loop may cause instability under linear state feedback. In particular, for coupled linear hyperbolic systems, recent contributions have highlighted the necessity of a change of paradigm in order to achieve delay-robust stabilization. The main contribution of this paper is to provide a new design for a state-feedback law for a PDE-ODE system that ensures the delay-robust stabilization. The original system can then be rewritten as a distributed delay equation for which it is possible to derive a stabilizing control law.

Problem Formulation

In this section we detail the notations used through this paper. For any integer $p > 0, \|.\|_{\mathbb{R}^p}$ is the classical euclidean norm on $\mathbb{R}^p$. We denote by $L^1([0,1],\mathbb{R})$, or $L^1([0,1])$ if no confusion arises, the space of real-valued functions defined on [0,1] whose absolute value is integrable. This space is equipped with the standard L^1 norm, that is, for any $f \in L^1([0,1])$

$$\| f \|_{L^1} = \int_0^1 |f(x)|\, dx.$$

We denote $L^2([0,1],\mathbb{R})$ the space of real-valued square-integrable functions defined on [0,1] with the standard L^2 norm, i.e., for any $f \in L^2([0,1]),\mathbb{R})$

$$\| f \|_{L^2}^2 = \int_0^1 f^2(x)\, dx.$$

The set $L^\infty([0,1],\mathbb{R})$ denotes the space of bounded real-valued functions defined on [0,1]with the standard L^∞ norm, i.e., for any $f \in L^\infty([0,1],\mathbb{R})$

$$\| f \|_{L^x} = sup_{x\in[0,1]}|f(x)|.$$

In the following, for $(u,v,X) \in \left(L^2([0,1])\right)^2 X\mathbb{R}^p$, we define the norm $\| (u,v,X) \| = \| u \|_{L^2} + \| v \|_{L^2} + \| X \|_{\mathbb{R}^p}$.

The set $c^p([0,1])$ stands for the space of real valued functions defined on [0,1] that are P times differentiable and whose Pth derivative is continuous. The set τ is defined as

$$\tau = \{(x,\xi) \in [0,1]^2\ s.t. \xi \le x\}.$$

$c(\tau)$ stands for the space of real-valued continuous functions on τ. For a positive real k and two reals $a < b$, a function f defined on $[a, b]$ is said to k-Lipschitz if for all $(x, y) \in [a, b]^2$, it satisfies $|f(x) - f(y)| \leq k|x - y|$. The symbol I_P (or I if no confusion arises) represents the pXp identity matrix. We use the notation $\hat{f}(s)$ for the Laplace transform of a function $f(t)$, provided it is well defined. The set $\mathcal{A}$ stands for the convolution Banach algebra of BIBO – stable generalized functions in the sense of Vidyasagar. A function g(.) belongs to $\mathcal{A}$ if it can be expressed as

$$g(t) = g_r(t) + \sum_{i=0}^{\infty} g_i\,\delta(t - t_i),$$

Where $g_r \in L^1(\mathbb{R}^+, \mathbb{R}), \sum_{i\geq 0} |g_i| < \infty, 0 = t_0 < t_1 < \cdots \, and\ \delta(.)$ is the direct distribution? The associated norm is

$$\parallel g \parallel_{\mathcal{A}} = \parallel g_r \parallel_{L^1} + \sum_{i\geq 0} |g_i|.$$

The set $\hat{\mathcal{A}}$ of Laplace transforms of elements in $\mathcal{A}$ is also Banach algebra with associated norm

$$\parallel \hat{g} \parallel_{\hat{\mathcal{A}}} = \parallel g \parallel_{\mathcal{A}}.$$

System Under Consideration

We consider a class of systems consisting of an ODE coupled to two hetero directional first-order linear hyperbolic systems in the actuation path. We consider systems of the form:

$$u_t(t, x) + \lambda u_x(t, x) = \sigma^{+-}(x) v(t, x)$$
$$v_t(t, x) - \mu v_x(t, x) = \sigma^{-+}(x) u(t, x)$$
$$\dot{X}(t) = AX(t) + Bv(t, 0),$$

Evolving in $\{(t, x) s.t\ t > 0, x \in [0,1]\}$, with the boundary conditions

$$u(t, 0) = qv(t, 0) + CX(t)$$
$$v(t, 1) = \rho u(t, 1) + U(t),$$

Where $X \in \mathbb{R}^p$ is the ODE state, $u(t, x) \in \mathbb{R}\ and\ v(t, x) \in \mathbb{R}$ are the PDE states and $U(t)$ is the control input. The in-domain coupling terms σ^{-+} and σ^{+-} belong to $C^0([0,1])$, the boundary coupling terms $q \neq 0$ (distal reflexion) and ρ (proximal reflexion), and the velocities λ and μ are constants. Furthermore, the velocities verify

$$-\mu < 0 < \lambda.$$

The initial conditions of the state (u, v) are denoted u_0 and v_0 and are assumed to belong to $L^1([0,1], \mathbb{R})$ and we consider only weak L^2 solutions to the system. The initial condition of the ODE is denoted X_0. Remark that this system naturally features several couplings that can be extended to the case $q = 0$ with a slight modification of the backstepping transformation.

Control Problem

The goal of this paper is to design a feedback control law $U = \mathcal{K}[(u, v, X)]$ where $\mathcal{K}: \left(L^2([0,1])\right)^2 X \mathbb{R}^p \rightarrow \mathbb{R}$ is a linear operator, such that:

The state (u, v, X) of the resulting feedback system exponentially converges to its zero equilibrium **(stabilization problem),** i.e. there exist $k_o \geq 0$ and $v > 0$ such that for any initial condition $(u_0, v_0, X_0) \in (L^1[0,1])^2 X \mathbb{R}^p$

$$\parallel (u, v, X) \parallel \leq k_0 e^{-vt} \parallel (u_0, v_0, X_0) \parallel, \quad t \geq 0.$$

The resulting feedback system is robustly stable with respect to small delays in the loop **(delay-robustness)**, i.e. there exists $\delta^* > 0$ such that for any $\delta \in [0, \delta^*]$, the control law $U(t - \delta)$ still stabilizes.

A control law that satisfies these two constraints is said to **delay-robustly stabilize** system

In this paper, we make the two following assumptions:

Assumption 1:

The pair (A, B) is stabilizable, i.e. there exists a matrix K such that $A + BK$ is Hurwitz.

Assumption 2:

The proximal reflection ρ and the distal reflection q satisfy $|\rho q| < 1$.

The first assumption is necessary for the stabilizability of the whole system, while the second assumption is required for the existence of a delay-robust linear feedback control. This second assumption is not restrictive since, if is not fulfilled, one could prove using arguments similar to those in Auriol that the open-loop transfer function has an infinite number of poles in the complex closed right half-plane. Consequently, one cannot find any linear state feedback law $U(.)$ that delay robustly stabilizes.

Lemma

Take A,B and K verifying Assumption 1 and any k such that holds. Then, the control law $\widetilde{U}_{ODE}(t) = K\left(e^{\frac{A}{\mu}}Y(t) + \int_{t-\frac{1}{\mu}}^{t} e^{A(t-v)}\, B\, \widetilde{U}_{ODE}(v)dv\right)$,

Exponentially stabilizes$Y(t)$. Furthermore, the state feedback

$$U_{ODE}(t) = \widetilde{U}_{ODE}(t) - (\rho - k)CX(t - \tfrac{1}{\lambda})$$

Exponentially stabilizes $X(t)$.

Proof

For the state-forecaster feedback $\widetilde{U}_{ODE}(.)$, the closed-loop system in

$$\dot{Y}(t) = AY(t) + B\widetilde{U}_{ODE}\left(t - \frac{1}{\mu}\right),\ t \geq \frac{1}{\mu} \text{ satisfies}$$

$$\dot{Y}(t) = (A + BK)Y(t), \quad t \geq \frac{1}{\mu}.$$

Exponential stability is guaranteed by the fact that $(A + BK)$ is Hurwitz. By construction of $Y(t)$ and using $\hat{\phi}(s) = 1 - (\rho - k)qe^{-\tau}$, we have that $X(t)$, solution of

$$\dot{X}(t) - (\rho - k)q\dot{X}(t - \tau) = AX(t) - (\rho - k)qAX(t - \tau) + (\rho - kBCXt-\tau + BUODEt-1\mu,$$ satisfies for any $t \geq \tau$,

$$X(t) = (\rho - k)qX(t - \tau) + Y(t).$$

Since $|(\rho - k)q| < 1 by\ |kq| + |\rho q| < 1$, $X(t)$ is also exponentially stable.

CONCLUSION

In this paper, a delay-robust stabilizing feedback control law was developed for a coupled hyperbolic PDE-ODE system. The proposed method combines using the back stepping approach with a second forecaster-type feedback. The second feedback control is obtained after a suitable change of variables that reduces the stabilization problem of the PDE-ODE system.

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