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Common Fixed Point Theorems Using R-Weakly Commuting Mappings

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ABSTRACT
In this paper, we prove a common fixed point theorem in modified intuitionistic fuzzy metric space by combining the ideas of point wise R- weak commutativity and reciprocal continuity of mappings satisfying contractive conditions. We also give example to prove validity of proved result.

***Key words:** Compatible maps; R-weakly commuting mappings; modified intuitionistic fuzzy metric space.*
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1. INTRODUCTION AND PRELIMINARIES:

Recently, R. Saadati et. al [10] introduced the modified intuitionistic fuzzy metric space and proved some fixed point theorems for compatible and weakly compatible maps. The paper [10] is the inspiration of a large number of papers [1-3, 6-9, 11] that employ the use of modified intuitionistic fuzzy metric space and its applications.

In this paper, we prove a common fixed point theorem in modified intuitionistic fuzzy metric space by combining the ideas of point wise *R*- weak commutativity and reciprocal continuity of mappings satisfying contractive conditions. We also give example to prove validity of proved result.

Firstly, we recall the following notions that will be used in the sequel.

Lemma 1.1 [4]:
Consider the set L^* and the operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2, x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2,$$

For every (x_1, x_2), (y_1, y_2) in L^*, Then $(L^*, \leq_{L^*})$ is a complete lattice.
We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

Definition 1.1 [5]:
A triangular norm (t-norm) on L^* is a mapping F : $(L^*)^2 \to L^*$ satisfying the following conditions:

1. F $(x, 1_{L^*}) = x$ for all x in L^*,
2. F (x, y) = F (y, x) for all x, y in L^*,
3. F $(x$, F $(y, z))$ = F $($F $(x, y), z)$ for all x, y, z in L^*,
4. If for all x, x', y, y' in L^*, $x \leq_{L^*} x'$ and $y \leq_{L^*} y'$ implies F $(x, y) \leq_{L^*}$ F (x', y').

Definition 1.2 [4,5]:
A continuous t-norm F on L^* is called continuous t – represent table if there exist a continuous t- norm $*$ and a continuous t – co norm $\Diamond$ on [0, 1] such that for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*[0,1]^2$, F $(x, y) = (x_1 * y_1, x_2 \Diamond y_2)$.

Definition 1.3 [10]:
Let M, N are fuzzy sets from $X^2 \times (0, +\infty) \to [0,1]$ such that $M(x, y, t) + N(x, y, t) \leq 1$ for all x, y in X and $t > 0$. The 3- tuple (X, $\zeta_{M,N}$, F) is said to be a modified intuitionistic fuzzy metric space if X is an arbitrary non empty set, F is a continuous t-represent table and $\zeta_{M,N}$ is a mapping $X^2 \times (0, +\infty) \to L^*$ satisfying the following conditions for every x, y in X and $t, s > 0$:

(a) $\zeta_{M,N}(x, y, t) >_{L^*} 0_{L^*}$;

(b) $\zeta_{M,N}(x, y, t) = 1_{L^*}$ if $x = y$;

(c) $\zeta_{M,N}(x, y, t) = \zeta_{M,N}(y, x, t)$;

(d) $\zeta_{M,N}$ $(x, y, t + s) \geq_{L^*}$ F $(\zeta_{M,N}$ (x, z, t), $\zeta_{M,N}$ $(z, y, s))$;

(e) $\zeta_{M,N}$ $(x, y, .)$: $(0,+\infty) \rightarrow L^*$ is continuous.

In this case, $\zeta_{M,N}$ is called a modified intuitionistic fuzzy metric. Here,
$\zeta_{M,N}$ $(X, y, t) = (M (x, y, t), N (x, y, t))$.

Remark1.1. [11]:
In a modified intuitionistic fuzzy metric space (*X*, $\zeta_{M,N}$, F), *M*(*x*, *y*, .) is non decreasing and *N*(*x*, *y*, .) is non- increasing for all *x*, *y* in *X*. Hence $\zeta_{M,N}$ (*x*, *y*, *t*) is non- decreasing with respect to *t* for all *x*, *y* in *X*.

Definition 1.4 [10]:
A sequence $\{x_n\}$ in a modified intuitionistic fuzzy metric space
(*X*, $\zeta_{M,N}$, F) is called a Cauchy sequence if for each $\varepsilon > 0$ and *t* > *0*, there exists $n_0 \in \square$ such that $\zeta_{M,N}(x_n, x_m, t) >_{L^*} (1-\varepsilon, \varepsilon)$ for each $n, m \geq n_0$ and for all *t*.

Definition 1.5 [10]:
A sequence $\{x_n\}$ in a modified intuitionistic fuzzy metric space
(*X*, $\zeta_{M,N}$, F) is said to be convergent to *x* in *X* , denoted by $x_n \rightarrow x$ if
$\lim_{n\to\infty} \zeta_{M,N}$ $(x_n, x, t) = 1_{L^*}$ for all *t*.

Definition 1.6 [10]:
A modified intuitionistic fuzzy metric space (*X*, $\zeta_{M,N}$, F) is said to be complete if every Cauchy sequence is converges to a point of it.

Definition 1.7 [10, 11]
A pair of self mappings *(f, g)* of modified intuitionistic fuzzy metric space (*X*, $\zeta_{M,N}$, F) is said to be compatible if $\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) = 1_{L^*}$ whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some *z* in *X*.

Definition 1.8 [11]
Two self-mappings *f* and *g* are called non-compatible if there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some *z* in *X* but either $\lim_{n\to\infty} \zeta_{M,N}(fgx_n, gfx_n, t) \neq 1_{L^*}$ or the limit does not exist for all z in X.

Definition 1.9 [11]:
A pair of self mappings *(f, g)* of a modified intuitionistic fuzzy metric space $(X, \zeta_{M,N}$, F) is said to be *R*-weakly commuting at a point *x* in *X* if $\zeta_{M,N}(fgx_n, gfx_n, t) \geq_{L^*} \zeta_{M,N}\left(fx_n, gx_n, t/R\right)$ for some *R* > 0.

Definition 1.10[11]:
The two self-maps *f* and *g* of a modified intuitionistic fuzzy metric space (*X*, $\zeta_{M,N}$, F) are called point wise *R*-weakly commuting on *X* if given *x* in *X* there exists *R* > 0 such that $\zeta_{M,N}(fgx, gfx, t) \geq_{L^*} \zeta_{M,N}\left(fx, gx, t/R\right)$.

2. Lemmas:
The proof of our result is based upon the following Lemmas:

Lemma2.1[8].
Let (*X*, $\zeta_{M,N}$, T) be modified intuitionistic fuzzy metric space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$,
$\zeta_{M,N}(x, y, kt) \geq_{L^*} \zeta_{M,N}(x, y, t)$.
Then *x* = *y*.

Lemma2.2[8].
Let (*X*, $\zeta_{M,N}$, T) be modified intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in *X*. If there exists a number $k \in (0,1)$ such that:
$\zeta_{M,N}(y_n, y_{n+1}, kt) \geq_{L^*} \zeta_{M,N}(y_{n-1}, y_n, t)$
For all $t > 0$ and *n* = 1, 2,3, ...
Then $\{y_n\}$ is a Cauchy sequence in *X*.

3. Main Results:
Lemma 3.1:
Let $(X, \zeta_{M,N}, T$) be a modified intuitionistic fuzzy metric space and let *(A, S)* and *(B, T)* be pairs of self mappings on *X* satisfying
(3.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,
(3.2) there exists a constant $k \in (0,1)$ such that

$\zeta_{M,N}(Ax,By,kt) \geq_{L^*} \min\{\zeta_{M,N}(Ty, By, t), \zeta_{M,N}(Sx,Ax,t), \zeta_{M,N}(Sx,By, \alpha t)\}$

For all $x, y \in X$, $t > 0$ and $\alpha \in (0,2)$. Then the continuity of one of the mappings in compatible pair *(A, S)* or *(B, T)* on $(X, \zeta_{M,N}, T)$ implies their reciprocal continuity.

Proof:
Proof of this result easily follows condition (3.2) and definition of compatible maps.

Theorem 3.1:
Let $(X, \zeta_{M,N}$, T) be a complete modified intuitionistic fuzzy metric space. Further, let *(A, S)* and *(B, T)* be point wise *R*- weakly commuting pairs of self mappings of *X* satisfying (3.1), (3.2). If one of the mappings in compatible pair *(A, S)* or *(B, T)* is continuous, then *A*, *B*, *S* and *T* have a unique common fixed point in *X*,

Proof:
Let $x_0 \in X$. By (3.1), we define the sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that for all *n* = 0, 1, 2 …

$y_{2n} = Ax_{2n} = Tx_{2n+1}, y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}.$

We show that $\{y_n\}$ is a Cauchy sequence in *X*. By (3.2) take $\alpha = 1-\beta, \beta \in (0,1)$, we have

$$\zeta_{M,N}(y_{2n+1}, y_{2n+2}, kt) = \zeta_{M,N}(Bx_{2n+1}, Ax_{2n+2}, kt) = \zeta_{M,N}(Ax_{2n+2}, Bx_{2n+1}, kt)$$
$$\geq_{L^*} \min\{\zeta_{M,N}(Tx_{2n+1}, Bx_{2n+1}, t), \zeta_{M,N}(Sx_{2n+2}, Ax_{2n+2}, t), \zeta_{M,N}(Sx_{2n+2}, Bx_{2n+1}, (1-\beta)t),\}$$
$$= \min\{\zeta_{M,N}(y_{2n}, y_{2n+1}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+2}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+1}, (1-\beta)t\}$$
$$= \min\{\zeta_{M,N}(y_{2n}, y_{2n+1}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+2}, t), 1_{L^*}\}$$
$$\geq_{L^*} \min\{\zeta_{M,N}(y_{2n}, y_{2n+1}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+2}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+2}, \beta t)\}$$

Taking $\beta \to 1$, we have

$$\zeta_{M,N}(y_{2n+1}, y_{2n+2}, kt) \geq_{L^*} \min\{\zeta_{M,N}(y_{2n}, y_{2n+1}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+2}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+2}, t)\}$$
$$\zeta_{M,N}(y_{2n+1}, y_{2n+2}, kt) \geq_{L^*} \min\{\zeta_{M,N}(y_{2n}, y_{2n+1}, t), \zeta_{M,N}(y_{2n+1}, y_{2n+2}, t)\}$$
$$\zeta_{M,N}(y_{2n+1}, y_{2n+2}, kt) \geq_{L^*} \zeta_{M,N}(y_{2n}, y_{2n+1}, t)$$

Similarly

$$\zeta_{M,N}(y_{2n+2}, y_{2n+3}, kt) \geq_{L^*} \zeta_{M,N}(y_{2n+1}, y_{2n+2}, t)$$

Therefore, for any *n* and *t*, we have

$$\zeta_{M,N}(y_n, y_{n+1}, kt) \geq_{L^*} \zeta_{M,N}(y_{n-1}, y_n, t).$$

Hence, by Lemma 2.2, $\{y_n\}$ is a Cauchy sequence in *X*. Since X is complete, $\{y_n\}$ converges to *z* in *X*. Its subsequences $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$ and $\{Sx_{2n+2}\}$ also converges to *z*.

Now, suppose that (*A*, *S*) is a compatible pair and *S* is continuous. Then by Lemma 2.1, *A* and S are reciprocally continuous, then $SAx_n \to Sz$, $ASx_n \to Az$ as $n \to \infty$. As, (*A*, *S*) is a compatible pair. This implies

$$\lim_{n\to\infty} \zeta_{M,N}(ASx_n, SAx_n, t) = 1_{L^*}$$
$$\zeta_{M,N}(Az, Sz, t) = 1_{L^*}.$$

Hence, $Az = Sz$.

Since $A(X) \subseteq T(X)$, there exists a point *p* in *X* such that $Az = Tp = Sz$.

By (3.2), take $\alpha = 1$,

$\zeta_{M,N}(Az, Bp, kt) \geq_{L^*} \min\{\zeta_{M,N}(Tp, Bp, t), \zeta_{M,N}(Sz, Az, t), \zeta_{M,N}(Sz, Bp, t),\}$

$\zeta_{M,N}(Az, Bp, kt) \geq_{L^*} \min\{\zeta_{M,N}(Az, Bp, t), \zeta_{M,N}(Az, Az, t), \zeta_{M,N}(Az, Bp, t),\}$

$\zeta_{M,N}(Az, Bp, kt) \geq_{L^*} \zeta_{M,N}(Az, Bp, t)$

Thus, by Lemma 2.1, we have *Az* = *Bp.*

Thus, *Az* = *Bp* =*Sz* = *Tp.*

Since, *A* and *S* are point wise *R* – weakly commuting mappings, there exists *R* > 0, such that

$\zeta_{M,N}(ASz, SAz, t) \geq_{L^*} \zeta_{M,N}\left(Az, Sz, t/R\right) = 1_{L^*}$

Therefore, *ASz* = *SAz* and *AAz* = *ASz* = *SAz* = *SSz.*

Similarly, *B* and *T* are point wise *R*-weakly commuting mappings, we have *BBp* = *BTp* = *TBp* = *TTp.*

Again by (3.2), take $\alpha = 1$,

$\zeta_{M,N}(AAz, Bp, kt) \geq_{L^*} \min\{\zeta_{M,N}(Tp, Bp, t), \zeta_{M,N}(SAz, AAz, t), \zeta_{M,N}(SAz, Bp, t),\}$

$\zeta_{M,N}(AAz, Az, kt) \geq_{L^*} \min\{\zeta_{M,N}(Tp, Tp, t), \zeta_{M,N}(AAz, AAz, t), \zeta_{M,N}(AAz, Az, t),\}$

$\zeta_{M,N}(AAz, Az, kt) \geq_{L^*} \zeta_{M,N}(AAz, Az, t)$

By Lemma 2.1, we have *AAz* = *A z* = *SAz.* Hence *Az* is common fixed point of *A* and *S.* Similarly by (3.2), *Bp* = *Az* is a common fixed point of *B* and *T.* Hence, *Az* is a common fixed point of *A, B, S* and *T.*

For Uniqueness:

We can easily prove uniqueness by using (3.2).

Corollary 3.1:

Let $(X, \zeta_{M,N}$ T) be a complete modified intuitionistic fuzzy metric space. Further, let *A* and *B* are reciprocally continuous mappings on *X* satisfying

(3.3) $\zeta_{M,N}(Ax, By, kt) \geq_{L^*} \min\{\zeta_{M,N}(y, By, t), \zeta_{M,N}(x, Ax, t), \zeta_{M,N}(x, By, \alpha t)\}$

For all $x, y \in X$, *t > 0* and $\alpha \in (0, 2)$ then pair *A* and *B* has a unique common fixed point.

Example 2.1:

Let *X* = [0, 20] and for each *t* > 0, define

$\zeta_{M,N}(x, y, t) = \left(\frac{t}{t+|x-y|}, \frac{|x-y|}{t+|x-y|}\right)$. Then $(X, \zeta_{M,N}$, T) is complete modified intuitionistic fuzzy metric space. Let *A, B, S* and *T* be self mappings of X defined as

A (2) = 2, *Au* = 3 if u > 0,
B (u) = 2 if *u* = 2 or u > 6, *Bu* = 6 if 0 < *u* ≤ 6,
S (2) = 2, *S* (*u*) = 6 if *u* > 0,
T (2) = 2, *T* (*u*) = 12 if 0 < *u* ≤ 6, *T* (*u*) = *u* – 3 if *u* > 6.

Then *A, B, S* and *T* satisfy all the conditions of above theorem with *k* ∈ (0, 1) and have a unique common fixed point *u* = 2.

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