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τ*-Generalized *b* Compactness and τ*-Generalized *b* Connectedness in Topological Spaces

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ABSTRACT
This paper deals with that τ^* -gb compact spaces and their properties are studied. The notion of τ^* -gb connectedness in topological spaces is also introduced and their properties are studied.

Keywords: *τ^*-gb closed sets, τ^* -gb compact space, τ^* -gb connectedness*

1. INTRODUCTION

The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness and connectedness. The productivity and fruitfulness of these notions of compactness and connectedness motivated mathematicians to generalize these notions.

D. Andrijevic [1] introduced a new class of generalized open sets in topological space called b-open sets. The class of b-open sets generates the same topology as the class of pre-open sets. Since the advent of this notion, several research paper with interesting results in different respects came into existence. M. Ganster and M. Steiner [5] introduced and studied the properties of gb closed sets in topological spaces. In 1970, Levine introduced the concept of generalized closed sets in topological spaces. Dunham[2] introduced the concept of the closure operator cl* and a new topology τ^* and studied some of their properties. A. Pushpalatha, S. Eswaran and P. RajaRubi[6] introduced a new class of sets called τ^* - generalized closed sets and studied some of their properties. The authors introduced the concepts of τ^* - generalized b- closed sets and contra τ^* - generalized b- continuous and studied some of their properties in topological spaces. Connectedness and Compactness are the most important and fundamental concepts in topology. The aim of this paper is to introduce the concept of τ^* - generalized b connectedness and τ^* - generalized b compactness in topological spaces.

2. Preliminaries

Throughout this paper (X, τ^*_1) and (Y, τ^*_2) represent the non – empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A \subseteq X$, the closure of A and interior of A with be denoted by cl(A) and int(A) respectively, union of all τ*-gb-open sets X contained in A is called τ*-gb- interior of A and it is denoted by τ*-gbint (A), the intersection of all τ*-gb- closed sets of X containing A is called τ*-gb closure of A and it is denoted by τ*-gb-c*l*(A).

Definition 2.1. [7] Let a subset A of a topological space (X,τ), is called a τ^* -generalized b- closed set (briefly τ^*-gb-closed) if $bcl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is τ^*-open in X.

Definition 2.2.[8] A function $f : (X, \tau_1^*) \rightarrow (Y, \tau_2^*)$ is called contra τ^*-generalized b -continuous if $f^{-1}(V)$ is τ^*-*gb* - closed in (X, τ_1^*) for every open set V in (Y, τ_2^*).

Definition 2.3.**[8]** A function $f : (X, \tau_1^*) \to (Y, \tau_2^*)$ is called τ^*-*gb* - continuous if the pre image of every open set of Y is τ^*-*gb* - open in X.

Remark 2.4.[7] Every b closed set is τ^* gb closed.

3. τ^* -gb-compactness

Definition 3.1: A collection $\{A_i : i \in \Lambda\}$ of τ^* -gb open sets in a topological space X is called a τ^* -gb open cover of a subset B of X if $B \subset \cup \{A_i : i \in \Lambda\}$ holds.

Definition 3.2: A topological space X is τ^* -gb compact if every τ^* -gb open cover of X has a finite sub cover.

Definition 3.3: A subset B of a topological space X is said to be τ^* -gb compact relative to X if, for every collection $\{A_i : i \in \Lambda\}$ of τ^* -gb open subsets of X such that $B \subset \cup \{A_i : i \in \Lambda\}$ there exists a finite subset Λ_0 of Λ such that $B \subseteq \cup \{A_i : i \in \Lambda_0\}$

Definition 3.4: A subset B of a topological space X is said to be τ^* -gb compact if B is τ^* -gb compact as a subspace of X.

Theorem 3.5: Every τ^* -gb closed subset of a τ^* -gb compact space is τ^* -gb compact relative to X.

Proof: Let A be a τ^* -gb closed subset of τ^* -gb compact space X. Then A^c is τ^* -gb open in X. Let $M = \{G_\alpha : \alpha \in \Lambda\}$ be a cover of A by τ^* -gb open sets in X. Then $M^* = M \cup A^c$ is a τ^* -gb open cover of X. Since X is τ^* -gb compact M^* is reducible to a finite sub cover of X, say $X = G_{\alpha 1} \cup G_{\alpha 2} \cup \ldots \cup G_{\alpha m} \cup A^c$, $G_{\alpha_k} \in M$. But A and A^c are disjoint hence $A \subset G_{\alpha_1} \cup \ldots\ldots\ldots\ldots \cup G_{\alpha_m}$, $G_{\alpha_k} \in M$ which implies that any τ^* -gb open cover M of A contains a finite sub cover. Therefore A is τ^* -gb compact relative to X.

Thus every τ^* -gb closed subset of a τ^* -gb compact space X is τ^* -gb compact.

Definition 3.6: A function $f: X \to Y$ is said to be τ^* -gb continuous if $f^{-1}(V)$ is τ^* -gb closed in X for every closed set V of Y.

Definition 3.7: A function $f: X \to Y$ is said to be τ^* -gb irresolute if $f^{-1}(V)$ is τ^* -gb closed in X for every τ^* -gb closed set V of Y.

Theorem 3.8: A τ^* -gb continuous image of a τ^* -gb compact space is compact.

Proof: Let $f: X \to Y$ be a τ^* -gb continuous map from a τ^* -gb compact space X onto a topological space Y. Let $\{A_i : i \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \Lambda\}$is a τ^* -gb open cover of X. Since X is a τ^* -gb compact it has a finite sub-cover say $\{ f^{-1}(A_1), f^{-1}(A_2) \ldots \ldots f^{-1}(A_n)\}$. Since f is onto $\{A_1 \ldots\ldots A_n\}$ is a cover of Y, which is finite. Therefore Y is compact.

Theorem 3.9: If a map $f: X \to Y$ is a τ^* -gb irresolute and a subset B of X is τ^* -gb compact relative to X, then the image f(B) is τ^* -gb compact relative to Y.

Proof: Let $\{A_\alpha : \alpha \in \Lambda\}$ be any collection of τ^* -gb open subsets of Y such that $f(B) \subset \cup \{A_\alpha : \alpha \in \Lambda\}$. Then $B \subset \cup \{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$holds. Since by hypothesis B is τ^* -gb compact relative to X there exists a finite subset Λ_0 of Λ such that $B \subset \cup \{ f^{-1}(A_\alpha) : \alpha \in \Lambda_0 \}$

Therefore we have $f(B) \subset \cup \{A_\alpha : \alpha \in \Lambda_0\}$ which shows that f(B) is τ^* -gb compact relative to Y.

Theorem 3.10: A space X is τ^* -gb compact if and only if each family of τ^* -gb closed subsets of X with the finite intersection property has a non-empty intersection.

Proof: Given a collection **A** of subsets of X, let C ={ X – A : A ∈ **A** } be the collection of their complements. Then the following statements hold.

(a) **A** is a collection of τ^* -gb open sets if and only if C is a collection of τ^* -gb closed sets.

(b) The collection **A** covers X if and only if the intersection $\cap_{c\varepsilon C} C$ of all the elements of C is non-empty.

(c) The finite sub collection $\{A_1 \ldots\ldots\ldots A_n\}$ of **A** covers X if and only if the intersection of the corresponding elements Ci = X – Ai of C is empty.

The statement (a) is trivial, while the (b) and (c) follow from De Morgan's law. $X - (D \cup_{\alpha \in j} A_\alpha) = D \cap_{\alpha \in j} (X - A_\alpha)$. The proof of the theorem now proceeds in two steps, taking the contrapositve positive of the theorem and then the complement.

The statement X is τ^* -gb compact equivalent to : Given any collection **A** of τ^* -gb open subsets of X, if

τ^* -gb **A** covers X, then some finite sub collection of **A** covers X. This statement is equivalent to its contrapositive, which is the following.

Given any collection **A** of τ^* -gb open sets , if no finite sub-collection of **A** covers X, then **A** does not cover X. Letting C be as earlier, the collection { X – A : A ∈ **A**} and applying (a) to (c), we see that this statement is in turn equivalent to the following.

Given any collection C of τ^* -gb closed sets, if every finite intersection of elements of C is non empty, then the intersection of all the elements of C is non empty. This is just the condition of our theorem.

4. τ^* -gb Connectedness

Definition 4.1: A topological space X is said to be τ^* -gb connected if X cannot be expressed as a disjoint union of two non empty τ^* -gb open sets. A subset of X is τ^* -gb connected if it is τ^* -gb connected as a subspace.

Example 4.2: Let X = { a,b} and let τ^* = { X, ∅,{a}} Then it is τ^* -gb connected.

Remark 4.3: Every τ^* -gb connected space is connected but the converse need not be true in general, which follows from the following example.

Example 4.4: Let X = {a,b} and τ^* = {X, ∅}. Clearly (X, τ^*) is connected. The τ^* -gb open sets of X are { X, ∅,{a}{b}}. Therefore (X, τ^*) is not a τ^* -gb connected space, because X = {a} ∪ {b} where {a} and {b} are non empty τ^* -gb open sets.

Theorem 4.5: For a topological space X the following are equivalent.

(i) X is τ^* -gb connected.

(ii) X and ∅ are the only subsets of X which are both τ^* -gb open and τ^* -gb closed.

(iii) Each τ^* -gb continuous map of X into a discrete space Y with at least two points is a constant map.

Proof: (i)=>(ii)

Let O be any τ^* -gb open and τ^* -gb closed subset of X. Then O^C is both τ^* -gb open and τ^* -gb closed. Since X is disjoint union of the τ^* -gb open sets O and O^C implies from the hypothesis of (i) that either O = ∅ or O = X

(ii)=>(i)

Suppose that X = A∪B where A and B are disjoint non empty τ^* -gb open subsets of X. Then A is both τ^* -gb open and τ^* -gb closed.

By assumption A = ∅ or X

Therefore X is τ^* -gb connected.

(ii)=>(iii)

Let f: X→ Y be a τ^* -gb continuous map. Then X is covered by τ^* -gb open and τ^* -gb closed covering {f^{-1}(Y) : y ∈ Y }

By assumption f^{-1}(y) = ∅ or X for each y ∈ Y. If f^{-1}(y) = ∅ for all y ∈ Y, then f fails to be a map. Then there exists only one point y ∈ Y such that f^{-1}(y) ≠ ∅ and hence f^{-1}(y) = X.

This shows that f is a constant map.

(iii)=>(ii)

Let O be both τ^* -gb open and τ^* -gb closed in X. Suppose O ≠ ∅. Let f: X→ Y be a τ^* -gb continuous map defined by f(O) = y and f(O^C)= {w} for some distinct points y and w in Y.

By assumption f is constant. Therefore we have O = X.

Theorem 4.6: If f: X→ Y is a τ^* -gb continuous and X is a τ^* -gb connected, then Y is connected.

Proof: Suppose that Y is not connected. Let Y = A∪B where A and B are disjoint non-empty open set in Y. Since f is τ^* -gb continuous and onto, X = f^{-1}(A) ∪ f^{-1}(B) where f^{-1}(A) and f^{-1}(B) are disjoint non-empty τ^* -gb open sets in X. This contradicts the fact that X is τ^* -gb connected. Hence Y is τ^* -gb connected

Theorem 4.7: If f: X→ Y is a τ^* -gb irresolute surjection and X is τ^* -gb connected, then Y is τ^* -gb connected.

Proof:

Suppose that Y is not τ^* -gb connected. Let Y = A∪B where A and B are disjoint non-empty τ^* -gb open set in Y. Since f is τ^* -gb irresolute and onto, X = f^{-1}(A)

$\cup$ $f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non empty τ^* -gb open sets in X. This contradicts the fact that X is τ^* -gb connected/ Hence Y is connected.

Theorem 4.8: In a topological space (X, τ) with at least two points, if bO(X, τ) = bCL(X, τ), then X is not τ^* -gb connected.

Proof: By hypothesis we have bO(X, τ) = bCL(X, τ) and by remark 2.4 we have every b closed set is τ^* -gb closed, there exists some non-empty proper subset of X which is both τ^* -gb open and τ^* -gb closed in X. So by last theorem 4.5 (i ,ii,iii) we have X is not τ^* -gb connected.

Definition 4.9: A topological space X is said to be $T_{\tau^*\text{-gb}}$ space if every τ^* -gb closed subset of X is closed subset of X.

Theorem 4.10: Suppose that X is a $T_{\tau^*\text{-gb}}$ space, then X is connected if and only if it is τ^* -gb connected if every τ^* -gb closed subset of X is closed subset of X.

Proof: Suppose that X is connected. Then X cannot be expressed as disjoint union of two non-empty proper subsets of X. Suppose X is not a τ^* -gb connected space.

Let A and B be any two τ^* -gb open subsets of X such that X = A $\cup$ B, where A $\cap$ B = $\emptyset$ and A $\subset$ X, B$\subset$ X. Since X is $T_{\tau^*\text{-gb}}$ - space and A,B are τ^* -gb open, A,B are open subsets of X, which contradicts that X is connected. Therefore X is τ^* -gb connected.

Conversely, every open set is τ^* -gb open. Therefore every τ^* -gb connected space is connected.

Theorem 4.11: If the τ^* -gb open sets C and D form a separation of X and if Y is τ^* -gb connected subspace of X, then Y lies entirely within C or D.

Proof: Since C and D are both τ^* -gb in X the sets C $\cap$ Y and D $\cap$ Y are τ^* -gb open in Y these two sets are disjoint and their union is Y. If they were both non-empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or in D.

Theorem 4.12: Let A be a τ^* -gb connected subspace of X. If A $\subset$ B $\subset$ τ^* -gbcl(A), then B is also τ^* -gb connected.

Proof: Let A be a τ^* -gb connected subspace of X. If A $\subset$ B $\subset$ τ^* -gbcl(A). Suppose that B = C $\cup$ D is a separation of B by τ^* -gb open sets. Then by theorem 4.11 above A must lie entirely in C or in D. Suppose that A $\subset$ C, then τ^* -gbcl(A) $\subseteq$ τ^* -gbcl(C). Since τ^* -gbcl(C) and D are disjoint, B cannot intersect D. This contradicts the fact that D is non-empty subset of B. So D = $\emptyset$ which implies B is τ^* -gb connected.

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