International Journal of Trend in Scientific Research and Development (IJTSRD)
International Open Access Journal
ISSN No: 2456 - 6470 | www.ijtsrd.com | Volume - 2 | Issue – 3

A Method for Solving Fractional Differential Equations

Prof. (Dr.) Deepak Raj
Panipat Institute of Engineering & Technology, Samalkha, Haryana, India

ABSTRACT
In the present paper, Bernstein operational matrix of fractional integration is developed and is applied for solving fractional differential equations of special type named Riccati differential equations. The validity and applicability of the proposed technique is illustrated through various particular cases which demonstrate their efficiency and simplicity as compare to the existing operational matrix techniques.

Keywords: Fractional differential equations, Operational matrix, Bernstein Polynomials

1. INTRODUCTION

Fractional calculus is a branch of mathematics that deals with generalization of the well-known operations of differentiations and integrations to arbitrary non-integers orders. Fractional derivative provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with the classical integer-order models, in which such effects are in fact neglected. The idea of modelling dynamic systems by fractional differential equations can be used in many fields of science and engineering including electrochemical process [1-2], dielectric polarization [3], earthquakes [4], fluid-dynamic traffic model [5], solid mechanics [6], bioengineering[7-9] and economics[10]. Fractional derivative and integrals also appear in theory of control of dynamical systems when control system and the controller are described by fractional differential equations.

In recent years, a number of methods have been proposed and applied successfully to approximate various types of fractional differential equations. The most used methods are Adomian decomposition method [11-13], Variational iteration method [14-15], Homotopy perturbation method [16-17], Homotopy analysis method [18], fractional differential transform method [19-23], power series method [24] and other methods [25-26].

Recently, wavelet based operational matrix has been also applied for the solution of the fractional differential equations. In 2010, Li et. al. [27] constructed Haar wavelet operational matrix of fractional integration with the use of Block pluse functions and successfully applied for getting solution of special type of fractional differential equation. In same year Li [28] and Li et al. [29] used another operational matrix based on Chebyshev wavelet and Haar wavelet respectively for the same problem. In 2011, Saadatmandi and Dehghan [30] used the concept of orthogonal polynomial and constructed for Legendre operational matrix of differentiation for solving such problems.

Bernstein polynomials have been used for solving numerically partial differential equations [31]. More recently, we have used Bernstein approximation for stable solution of problem of Abel inversion [32-33], generalized Abel integral equations arising in classical theory of Elasticity [34] and Lane-Emden equations [35].

The aim of this paper is to present an efficient numerical method for solving fractional differential equations of special type called Riccati differential equations. In the second section, Bernstein polynomials are defined and in third section Bernstein operational matrix of fractional integration is constructed. Finally, the constructed operational matrix is used to solve some special type of nonlinear fractional differential equations.

2. BERNSTEIN POLYNOMIALS AND FUNCTION APPROXIMATIONS

2.1. Bernstein polynomials

The Bernstein polynomial, named after Sergei Natanovich Bernstein, is a polynomial in the Bernstein form that is a linear combination of Bernstein basis polynomials. The Bernstein basis polynomials of degree n are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \ , \quad for\ i = 0,1,2,\cdots,n \cdot \qquad (1)$$

There are $(n+1)$ n^{th} degree Bernstein basis polynomials forming a basis for the linear space V_n consisting of all polynomials of degree less than or equal to n in **R**[x]-the ring of polynomials over the field **R.** For mathematical convenience, we usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$. Any polynomial $B(x)$ in V_n may be written as

$$B(x) = \sum_{i=0}^{n} \beta_i B_{i,n}(x) \cdot \qquad (2)$$

The coefficients β_i are called Bernstein or Bezier coefficients. We will follow this convention as well. These polynomials have the following properties:

(i) $B_{i,n}(0) = \delta_{i0}$ and $B_{i,n}(1) = \delta_{in}$, where δ is the Kronecker delta function.

(ii) $B_{i,n}(t)$ has one root, each of multiplicity i and $n-i$, at $t = 0$ and $t = 1$ respectively.

(iii) $B_{i,n}(t) \geq 0$ for $t \in [0,1]$ and $B_{i,n}(1-t) = B_{n-i,n}(t)$.

Function approximation

If A function $f \in L^2[0,1]$ may be written as

$$f(t) = \lim_{n\to\infty} \sum_{i=0}^{n} c_{in} b_{in}(t) \ , \qquad (3)$$

where, $c_{in} = \langle c, b_{in} \rangle$ and $\langle , \rangle$ is the standard inner product on $L^2[0,1]$.

If the series (5) is truncated at $n = m'$, then we have

$$f \cong \sum_{i=0}^{m} c_{im} b_{im} = C^T \Psi(t), \qquad (4)$$

where, C and $\Psi(t)$ are $(m'+1)\times 1$ matrices given by

$$C = [c_{0m}, c_{1m}, \cdots, c_{mm}]^T, \text{ and} \qquad (5)$$

$$\Psi(t) = [b_{0m'}(t), b_{1m'}(t), \cdots, b_{m'm'}(t)]^T \cdot \qquad (6)$$

For taking the collocation points, let t_0 be any point near to zero and other point as following:

$$t_i = t_0 + \frac{i}{m'+1} i > 0. \qquad (7)$$

Let us use the notation $m = m' + 1$, for defining the Bernstein polynomial matrix $\Phi_{m\times m}$ as follows:

$$\psi(t) \triangleq [B_{0m'}(t_0), B_{1m'}(t_1), \ldots\ B_{m'm'}(t_{m'})] \quad (8)$$

$$\Phi_{m\times m} = \begin{bmatrix} 0.0060 & 0.4019 & 0.2622 & 0.0930 & 0.0162 & 0.0006 \\ 0.0000 & 0.2024 & 0.3292 & 0.2334 & 0.0816 & 0.0079 \\ 0.0000 & 0.0544 & 0.2205 & 0.3125 & 0.2185 & 0.0528 \\ 0.0000 & 0.0082 & 0.0830 & 0.2353 & 0.3292 & 0.1995 \\ 0.0000 & 0.0007 & 0.0167 & 0.0945 & 0.2646 & 0.4019 \\ 0.0000 & 0.0000 & 0.0014 & 0.0158 & 0.0886 & 0.3373 \end{bmatrix}$$

3. BERNSTEIN OPERATIONAL MATRIX OF FRACTIONAL INTEGRATION

3.1. Block Pulse Functions and operational matrix of fractional integration

A set of Block Pulse Functions (BPF) is defined on [0,1) as:

$$b_i(t) = \begin{cases} 1 & i/m \le t < i+1/m, \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where i = 0,1,…m-1.

The functions $b_i(t)$ are disjoint and orthogonal. That is

$$b_i(t)b_j(t) = \begin{cases} 0 & i \neq j \\ b_i & i = j \end{cases} \tag{10}$$

$$\int_0^1 b_i(t)b_j(t)\,dt = \begin{cases} 0 & i \neq j \\ 1/m & i = j \end{cases} \tag{11}$$

Kilicman and Al Zhour [36] have obtained the Block Pulse operational matrix of the fractional integration F^{α} as following:

$$(I^{\alpha}B_m)(t) \approx F^{\alpha}B_m(t). \tag{12}$$

Where $B_m(t) \triangleq [b_0(t), b_1(t), \dots, b_{m-1}(t)]^T$ and

$$F^{\alpha} = \frac{1}{m^{\alpha}}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{m-1} \\ 0 & 1 & \varepsilon_1 & \dots & \varepsilon_{m-2} \\ 0 & 0 & 1 & \dots & \varepsilon_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

with $\varepsilon_k = (k+1)^{\alpha+1} 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

3.2. Bernstein operational matrix of the fractional integration

To derive the Bernstein polynomials operational matrix of the fractional integration, we rewrite Riemann–Liouville fractional integration as follows

$$(I^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau$$

$$= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \tag{13}$$

where $\alpha \in R$ is the order of the integration, $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1} * f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$.

In general the operational matrix of integration of the vector $\psi(t)$ defined in equation (8) can be obtained as

$$\int_0^t \psi(\tau)d\tau \approx P\,\psi(t). \tag{14}$$

Where, P is the $m \times m$ Operational matrix for integration.

Bernstein polynomials can also be expanded and approximated into an m-term block pulse functions (BPF) as $\psi_m(t) = \Phi_{m\times m}B_m(t)$, where

$$B_m(t) \triangleq [b_0(t), b_1(t), \dots, b_{m-1}(t)]^T. \tag{15}$$

Let us consider, the matrix $P^{\alpha}_{m\times m}$ is the Bernstein polynomials operational matrix of the fractional integration then

$$(I^{\alpha}\psi_m)(t) \approx P^{\alpha}_{m\times m}\psi_m(t). \tag{16}$$

Now, from equations (13) and (16) we have

$$(I^{\alpha}\psi_m)(t) \approx (I^{\alpha}\Phi_{m\times m}B_m)(t) = \Phi_{m\times m}(I^{\alpha}B_m)(t) \approx \Phi_{m\times m}F^{\alpha}B_m(t). \tag{17}$$

From equations (16) and (17) we get

$$P^{\alpha}_{m\times m}\psi_m(t) = P^{\alpha}_{m\times m}\Phi_{m\times m}B_m(t) = \Phi_{m\times m}F^{\alpha}B_m(t).$$

Then, the Bernstein polynomials operational matrix of the fractional integration $P^{\alpha}_{m\times m}$ is given by

$$P^{\alpha}_{m\times m} = \Phi_{m\times m} F^{\alpha}\, \Phi_{m\times m}{}^{-1}. \qquad (18)$$

4. RESULTS AND DISCUSSIONS

In this section, the implementation of the proposed Bernstein Operational matrix of integration is given to solve linear and nonlinear fractional order differential equation. Some well-known examples with known exact solution (and other numerical solutions) have been considered to show the validity of the presented operational matrix technique. For each of the stated examples comparison and discussions have been done.

Examples 4.1 Consider the differential equation from Li [28],

$$D^{\alpha} y(t) \ = \ -[y(t)]^2 \ + 1, \ \ 0 < \ \alpha \leq \ 1 \ , \ 0 \leq t \ , \qquad (19)$$

subject to the initial condition y(t)=0, having exact solution for

$\alpha = 1$ is $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$ and we can observe that, as $t \to \infty, y(t) \to 1$.

Let $D^{\alpha} y(t) = \ K_m^T \ \psi_m\,(t)$ then

$$y(t) = \ K_m^T \ P^{\alpha}_{m\times m} \psi_m\,(t) \qquad (20)$$

Using equation (15) we have

$$y(t) = \ K_m^T \ P^{\alpha}_{m\times m} \Phi_{m\times m} B_m\,(t). \qquad (21)$$

Assume $K_m^T \ P^{\alpha}_{m\times m} \Phi_{m\times m} = [a_1, a_2, \ldots \ldots \ldots \ldots a_m]$, we have

$$[y(t)]^2 = [a_1 b_1(t) + a_2 b_2(t) + .. \ . a_m b_m(t)]^2 = [a_1^2, a_2^2, \ldots a_m^2] B_m\,(t) \qquad (22)$$

From Equations (27) - (29) and Eq. (26), we have

$$K_m^T \ \Phi_{m\times m} B_m\,(t) + [a_1^2, a_2^2, \ldots a_m^2] B_m(t) - [1, \ldots 1] B_m\,(t) = 0 \qquad (23)$$

For solving the system of nonlinear algebraic equations eqs. (23), we have used the Matlab function fsolve. The obtained results with the exact solution are discussed as follows:

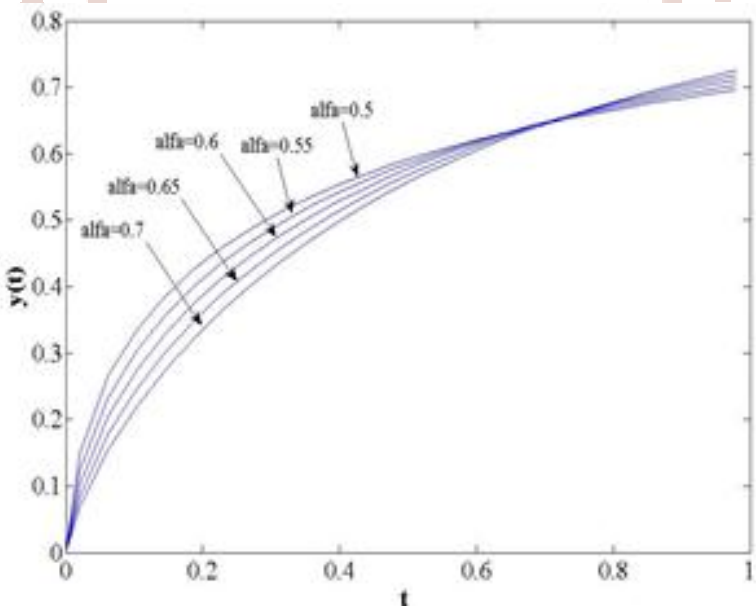

Fig. 4.1.1 Plot of Bernstein solutions of example 1 for different α=0.5, 0.55, 0.6, 0.65,0.7 for m=24.

Table 1: Proposed solution and other solution for example 4.1.1

t	Proposed method $\alpha = 0.75$	Ref. [16] $\alpha = 0.75$	Proposed method $\alpha = 1$	Ref. [16] $\alpha = 1$	Exact $\alpha = 1$
0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.18926	0.184795	0.09966	0.099668	0.099668
0.2	0.30934	0.313795	0.19731	0.197375	0.197375
0.3	0.40421	0.414562	0.29121	0.291312	0.291313
0.4	0.48147	0.492889	0.37994	0.379944	0.379949

0.5	0.54480	0.462117	0.46194	0.462078	0.462117
0.6	0.59761	0.597393	0.53695	0.536857	0.537050
0.7	0.64160	0.631772	0.60410	0.603631	0.604368
0.8	0.6785	0.660412	0.66387	0.661706	0.664037
0.9	0.7101	0.687960	0.71628	0.709919	0.716298
1.0	0.73355	0.718260	0.76141	0.746032	0.761594

Examples 4.2 Consider the differential equation from Li et al. [29],
$D^{\alpha}y(t) = 2y(t) - [y(t)]^2 + 1, 0 < \alpha \leq 1, 0 \leq t < 5$, (24)

subject to the initial condition y(t)=0, having exact solution for $\alpha = 1$ is $y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}\,t + \frac{1}{2}\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right)$ and we can observe that, as $t \to \infty, y(t) \to 1 + \sqrt{2}$.

Let $D^{\alpha}y(t) = K_m^T\ \psi_m(t)$ then
$y(t) = K_m^T\ P_{m\times m}^{\alpha}\psi_m(t)$ (25)

Using equation (15) we have
$y(t) = K_m^T\ P_{m\times m}^{\alpha}\Phi_{m\times m}B_m(t)$. (26)

Assume$K_m^T\ P_{m\times m}^{\alpha}\Phi_{m\times m} = [a_1, a_2, \dots \dots \dots \dots a_m]$, we have
$[y(t)]^2 = [a_1b_1(t) + a_2b_2(t) + \dots\ . a_mb_m(t)]^2 = [a_1^2, a_2^2, \dots a_m^2]B_m(t)$ (27)

From Equations (25) - (26) and Eq. (27), we have
$K_m^T\,\Phi_{m\times m}B_m(t) + [a_1^2, a_2^2, \dots a_m^2]B_m(t) - 2K_m^T\ P_{m\times m}^{\alpha}\Phi_{Bm\times m}B_m(t) - [1, \dots 1]B_m(t) = 0$ (28)

For solving the system of nonlinear algebraic equations eqs. (28), we have used the Matlab function fsolve. The obtained results with the exact solution are discussed as follows:

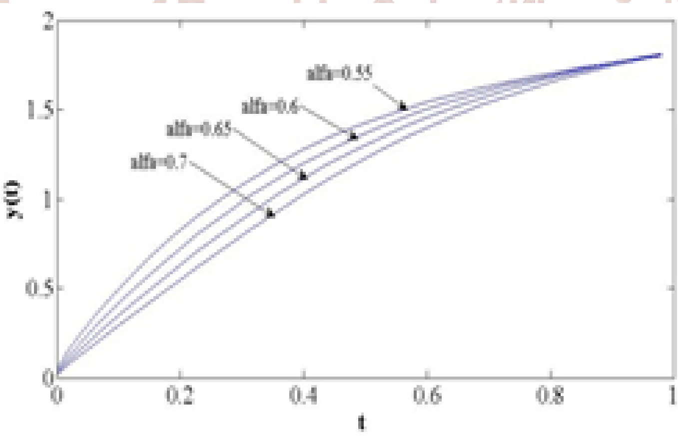

Fig.4.2.1 Bernstein solution for different α=0.55, 0.6, 0.65,0.7 for m=24.

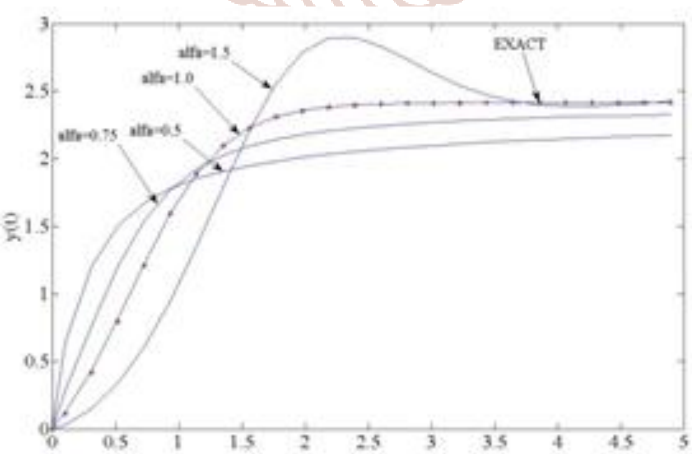

Fig.4.2.2 Bernstein solution for different α=0.5, 0.75, 1.0,1.5 and exact for m=24.

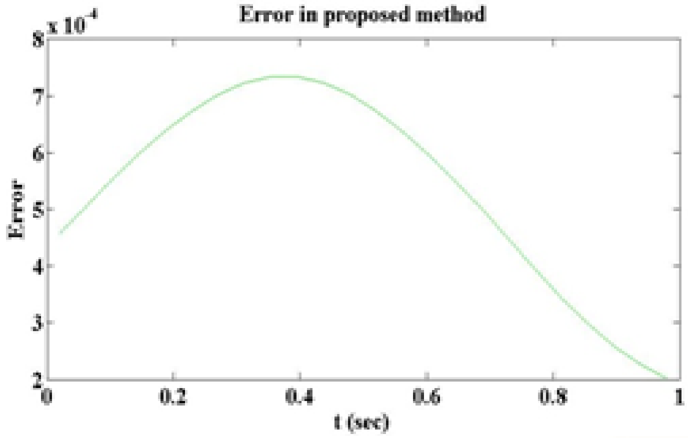

Fig.4.2.3 Absolut error for different for m=24

Examples 4.3 Consider the fractional nonlinear differential equation from Li et al. [29],

$$D^{\alpha}y(t) = [y(t)]^2 + 1, \ \ 0 < \alpha \leq 1, \ 0 \leq t, \tag{29}$$

subject to the initial condition y(t)=0.

Let $D^{\alpha}y(t) = K_m^T \psi_m(t)$ then $y(t) = K_m^T P_{m\times m}^{\alpha}\psi_m(t)$ (30)

Using equation (15) we have

$$y(t) = K_m^T P_{m\times m}^{\alpha}\Phi_{m\times m}B_m(t). \tag{31}$$

Assume $K_m^T P_{m\times m}^{\alpha}\Phi_{m\times m} = [a_1, a_2, \dots \dots \dots a_m]$, we have

$$[y(t)]^2 = [a_1b_1(t) + a_2b_2(t) + \dots a_mb_m(t)]^2 = [a_1^2, a_2^2, \dots a_m^2]B_m(t) \tag{32}$$

From Equations(29) - (31) and Eq. (32), we have

$$K_m^T \Phi_{m\times m}B_m(t) - [a_1^2, a_2^2, \dots a_m^2]B_m(t) - [1, \dots 1]B_m(t) = 0 \tag{33}$$

For solving the system of nonlinear algebraic equations eqs. (33), we have used the Matlab function f solve. The obtained results with the exact solution is disscussed as follows numerical solution for m=24.

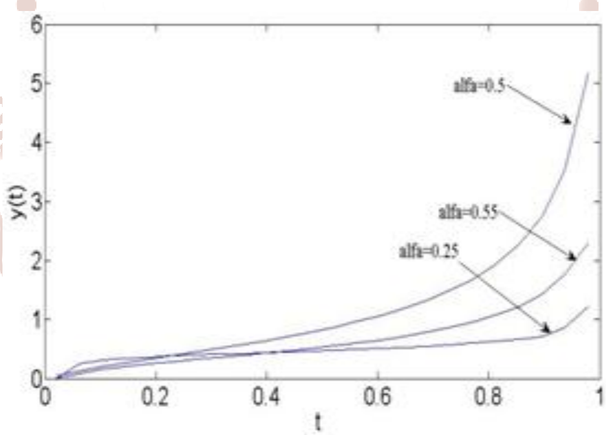

Fig.4.3.1 Bernstein solution for different α=0.25, 0.55, 0.5 for m=24.

Table 4.3.3 Proposed solution and other solution for $\alpha = 1.5$

x	ADM[13]	FDTM[19]	REF[29]	PROPOSED
0.1	0.023790	0.023790	0.023869	0.024069
0.3	0.123896	0.123896	0.123917	0.124178
0.5	0.268856	0.268856	0.268906	0.269164
0.7	0.453950	0.453950	0.454000	0.454293
0.9	0.685056	0.685056	0.685117	0.685388
1.0	0.822511	0.822509	0.822617	0.822610

CONCLUSIONS

First, we have derived the Bernstein operational matrix of fractional order integration and then applied it for solving the fractional Riccati differential equation. The advantage of the proposed operational matrix method over others is that only small size operational matrix is required to provide the solution at high accuracy. The solved illustrative examples demonstrate the efficiency and simplicity of the proposed method compared with the existing ones.

REFERENCES

[1] Ichise, M., Nagayanagi, Y. and Kojima, T. (1971). An analog simulation of non-integer order transfer functions for analysis of electrode processes. *J. Electronical. Chem. Interfacial Electrochem.* 33, 253–265.

[2] Sun, H. H., Onaral, B. and Tsao, Y. (1984). Application of positive reality principle to metal electrode linear polarization phenomena. *IEEE Trans. Biomed. Eng. BME*-31 10, 664–674.

[3] Sun, H. H., Abdelwahab, A. A. and Onaral, B. (1984). Linear approximation of transfer function with a pole of fractional order. *IEEE Trans. Automat. Control. AC*-29 (5), 441–444.

[4] He, J. H., (1998). Nonlinear oscillation with fractional derivative and its applications, in. *International Conference on Vibrating Engineering'98, Dalian, China*. 288-291.

[5] He, J. H., (1999). Some applications of nonlinear fractional differential equations and their approximations. *Bull. Sci. Technol.* 15 (2), 86-90.

[6] Rossikhin, Y. A. and Shitikova, M. V. (1997). Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. *Appl. Mech. Rev.* 50, 15-67.

[7] Magin R. L., (2004) Fractional calculus in bioengineering. *Crit. Rev. Biomed. Eng.* 32 (1), 1-104.

[8] Magin, R. L. (2004). Fractional calculus in bioengineering-part 2. *Crit. Rev. Biomed. Eng.* 32 (2), 105-193.

[9] Magin, R. L. (2004). Fractional calculus in bioengineering-part 3. *Crit. Rev. Biomed. Eng.* 32 (3/4), 194-377.

[10] Baillie, R. T. (1996). Long memory processes and fractional integration in econometrics. *J. Econometrics.* 73, 5-59.

[11] Odibat,Z. And Momani S. (2008). Numerical methods for nonlinear partial differential equations of fractional order. *Appl. Math. Model.* 32, 28–39.

[12] MomaniS. and Odibat, Z., (2007) Numerical approach to differential equations of fractional order. *J. Comput. Appl. Math.* 207, 96–110.

[13] Shawagfeh N. T., (2002) Analytical approximate solutions for fractional differential equations. *Appl. Math. Comput.* 131, 517-529.

[14] Sweilam, N. H., Khader M. M., Al-Bar, R. F., (2007). Numerical studies for a multi-order fractional *differential equation, Phys. Lett.* A 371, 26–33.

[15] Das, S. (2009). Analytical solution of a fractional diffusion equation by variational iteration method. *Comput. Math. Appl.* 57, 483–487.

[16] Momani, S. and Odibat, Z., (2008). Modified Homotopy perturbation method: Application to quadratic Riccati differential equations with fractional order. *Chaos Solitons and Fractals* 36, 167-174.

[17] Sweilam, N. H., Khader, M. M. and Al-Bar, R. F., (2007). Numerical studies for a multi-order fractional differential equation. *Phys. Lett.* A 371, 26-33.

[18] Hashim, I., Abdulaziz, O. and Momani,S., (2009). Homotopy analysis method for fractional IVPs, Commun. *Nonlinear Sci. Numer. Simul.* 14, 674-684.

[19] Arikoglu, A. and Ozkol, I., (2007). Solutionof fractional differential equations by using differential transform method. Chaos, *Solitons Fract.* 34, 1473–1481.

[20] Arikoglu, A. and Ozkol, I., (2009). Solution of fractional integro-differential equations by using fractional differential transform method. *Chaos, Solitons Fract.* 405, 21–529.

[21] Darania, P. and Ebadian, A., (2007). A method for the numerical solution of the integro-differential equations. *Appl. Math. Comput.* 188, 657–668.

[22] Erturk, V. S. and Momani, S., (2008). Solving systems of fractional differential equations using differential transform method. *J. Comput. Appl. Math.* 215, 142–151.

[23] Erturk, V. S., Momani, S. and Odibat, Z., (2008). Application of generalized differential transforms method to multi-order fractional differential equations. *Comm. Nonlinear Sci. Numer. Simulat*. 13, 1642–1654.

[24] Odibat, Z. and Shawagfeh, N., (2007). Generalized Taylor's formula. *Appl. Math. Comput.* 186, 286–293.

[25] Kumar P. and Agrawal, O. P., (2006). An approximate method for numerical solution of fractional differential equations. *Signal Processing* 86, 2602-2610.

[26] Liua, F., Anh, V. and Turner I., (2004). Numerical solution of the space fractional Fokker-Planck equation. *J. Comput. Appl. Math.* 166, 209-219.

[27] Li, Y. and Zaho, W., (2010). Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. *appl. Math. Comput.* 216, 2276-2285.

[28] Li, Y., (2010). Solving a nonlinear fractional differential equation using Chebyshev wavelets. *Communications in Nonlinear Science and Numerical Simulation*, 15 (9), 2284-2292.

[29] Li, Y. and Hu L. Solving fractional Riccati differential equations using Haar wavelet. 2010 Third International conference on Information and Computing.

[30] Saadatmandi, A. and Dehghan, M., (2010). A new operational matrix for fractional-order differential equations. *Comput. Math. Appl.*, 59 (3), 1326-1336.

[31] Bhatta, D. (2008). Use of modified Bernstein polynomials to solve Kdv-Burgers equation numerically. *Appl. Math. Comput.*, 206, 457-464.

[32] Mandal, B. N. and Bhattacharya, S., (2007). Numerical solution of some classes of integral equations using Bernstein polynomials. *Appl. Math. Comput.* 109, 1707-1716.

[33] Singh, O. P., Singh, V. K. and Pandey, R. K., (2010). A stable numerical inversion of Abel's integral equations using almost Bernstein operational matrix. *J. Quantitative Spectroscopy & Radiative Transfer*, 111, 245-252.

[34] Singh, V. K., Pandey, R. K. and Singh, O. P., (2009). New stable numerical solution singular integral equation of Abel type using normalized Bernstein polynomials. *Applied Math. Sciences*, 3, 241-255.

[35] Pandey, R. K. and Mandal, B. N., (2010). Numerical solution of system of generalized Abel integral equations using Bernstein's polynomials. *Journal of Advance Research in scientific Computing*, 2, 44-53.

[36] Pandey, R. K. and Kumar, N., (2012). Solution of Lane-Emden type equations using Bernstein operational matrix of differentiation. *New Astronomy*, 17, 303-308.

[37] Kilicman, A. and Al Zhour ZAA. (2007). Kronecker operational matrices for fractional calculus and some applications. *Appl. Math. Comput*. 187(1), 250–265.