CRACK PROBLEMS CONCERNING BOUNDARIES OF CONVEX LENS LIKE FORMS

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Abstract

The singular stress problem of a peripheral edge crack around a cavity of spherical portion in an infinite elastic medium when the crack is subjected to a known pressure is investigated. The problem is solved by using integral transforms and is reduced to the solution of a singular integral equation of the first kind. The solution of this equation is obtained numerically by the method due to Erdogan, Gupta , and Cook, and the stress intensity factors are displayed graphically.

Also investigated in this paper is the penny-shaped crack situated symmetrically on the central plane of a convex lens shaped elastic material.

Key words: cavity of spherical portion/ peripheral edge crack/penny-shaped crack /SIF.

1.Introduction.

The problem of determining the distribution of stress in an elastic medium containing a circumferential edge crack has been investigated by several researchers including the present author. Among these investigations, the notable ones are Keer et al.[1,2],Atsumi and Shindo[3,4], and Lee[5,6]. Keer et al.[1] considered a circumferential edge crack in an extended elastic medium with a cylindrical cavity the analysis of which provides immediate application to the study of cracking of pipes and nozzles if the crack is small.

Another important problem involving a circumferential edge crack is that concerned with a spherical cavity. Atsumi and Shindo[4] investigated the singular stress problem of a peripheral edge crack around a spherical cavity under uniaxial tension field. In more recent years, Wan et al.[7] obtained the solution for cracks emanating from surface semi-spherical cavity in finite body using energy release rate theory. In previous studies concerning the spherical cavity with the circumferential edge crack, the cavity was a full spherical shape. In this present analysis, we are concerned with a cavity of a spherical portion, rather than a full spherical cavity. More briefly describing it, the cavity looks like a convex lens. Here, we employ the known methods of previous investigators to derive a singular integral equation of the first kind which was solved numerically, and obtained the s.i.f. for various spherical portions. It is also shown that when this spherical portion becomes a full sphere, the present solution completely agrees with the already known solution.

2.Formulation of problem and reduction to singular integral equation. [0]

We employ cylindrical coordinates (r, ϕ, z) with the plane $z = 0$ coinciding the plane of peripheral edge crack. The spherical coordinates

(ρ, θ, ϕ) are connected with the cylindrical coordinates by

$$z = \rho \cos\theta, \qquad r = \rho \sin\theta.$$

Spherical coordinates (ζ, ϑ, ϕ) whose origin is located at $z = -\delta$, $r = 0$, and is the center of the upper spherical surface, are also used. The cavity is symmetrical with respect to the plane $z = 0$.

The crack occupies the region $z = 0$, $1 \leq r \leq \gamma$. So the radius of the spherical cavity is $\zeta_0 = \sqrt{1 + \delta^2}$.

The boundary conditions are:

On the plane $z = 0$, we want the continuity of the shear stress, and the normal displacement:

$$u_z(r, 0^+) - u_z(r, 0^-) = 0, \qquad \gamma \leq r < \infty, \quad (2.1)$$

$$\sigma_{rz}(r, 0^+) - \sigma_{rz}(r, 0^-) = 0, \qquad 1 \leq r < \infty. \quad (2.2)$$

And the crack is subjected to a known pressure $p(r)$, i.e.,

$$\sigma_{zz}(r, 0^+) = -p(r), \qquad 1 \leq r \leq \gamma. \quad (2.3)$$

On the surface of the spherical cavity, stresses are zero:

$$\sigma_{\zeta\zeta}(\zeta_0, \vartheta) = 0, \qquad (2.4)$$

$$\sigma_{\zeta\vartheta}(\zeta_0, \vartheta) = 0. \qquad (2.5)$$

We can make use of the axially symmetric solution of the equations of elastic equilibrium due to Green and Zerna [8] which states that if $\varphi(r, z)$ and $\psi(r, z)$ are axisymmetric solutions of Laplace equation, then the equations

$$2\mu u_r = \frac{\partial \varphi}{\partial r} + z \frac{\partial \psi}{\partial r}, \qquad (2.6)$$

$$2\mu u_z = \frac{\partial \varphi}{\partial z} + z \frac{\partial \psi}{\partial z} - (3 - 4\nu)\psi, \qquad (2.7)$$

where μ is the modulus of rigidity and ν is Poisson's ratio, provide a possible displacement field. The needed components of stress tensor are given by the equations

$$\sigma_{rz} = \frac{\partial^2 \phi}{\partial r \partial z} + z \frac{\partial^2 \psi}{\partial r \partial z} - (1 - 2\nu)\frac{\partial \psi}{\partial r}, \qquad (2.8)$$

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial z^2} + z \frac{\partial^2 \psi}{\partial z^2} - 2(1 - \nu)\frac{\partial \psi}{\partial z}. \qquad (2.9)$$

The functions $\phi^{(1)}$ and $\phi^{(2)}$ for the regions $z > 0$ and $z < 0$, respectively, are chosen as follows:

$$\phi^{(1)}(r, z) = (2\nu - 1)\int_0^\infty \xi^{-1} A(\xi) J_0(\xi r) e^{-\xi z} d\xi$$

$$+ \sum_{n=0}^\infty a_n \frac{P_n(\cos\theta)}{\rho^{n+1}}, \qquad (2.10)$$

$$\phi^{(2)}(r, z) = (2\nu - 1)\int_0^\infty \xi^{-1} A(\xi) J_0(\xi r) e^{\xi z} d\xi$$

$$- \sum_{n=0}^\infty a_n(-1)^n \frac{P_n(\cos\theta)}{\rho^{n+1}}. \qquad (2.11)$$

Here the superscripts (1) and (2) are taken for the region $z > 0$ and $z < 0$, respectively. The functions $\psi^{(1)}$ and $\psi^{(2)}$ are chosen as follows:

$$\psi^{(1)}(r, z) = \int_0^\infty A(\xi) J_0(\xi r) e^{-\xi z} d\xi$$

$$+ \sum_{n=0}^\infty b_n \frac{P_n(\cos\theta)}{\rho^{n+1}}, \qquad (2.12)$$

$$\psi^{(2)}(r, z) = -\int_0^\infty A(\xi) J_0(\xi r) e^{\xi z} d\xi$$

$$+ \sum_{n=0}^\infty b_n(-1)^n \frac{P_n(\cos\theta)}{\rho^{n+1}}. \qquad (2.13)$$

Then we can immediately satisfy condition (2.2) by this choice of functions (2.10)-(2.13).

Now the condition (2.1) requires

$$\int_0^\infty A(\xi) J_0(\xi r) d\xi = 0, \quad r > \gamma. \qquad (2.14)$$

Equation (2.14) is automatically satisfied by setting

$$A(\xi) = \int_1^\gamma t g(t) J_1(\xi t) dt. \qquad (2.15)$$

Then from the boundary condition (2.3), we obtain

$$\int_0^\infty \xi A(\xi) J_0(\xi r) d\xi - \sum_{n=0}^\infty a_{2n} \frac{(2n+1)^2 P_{2n}(0)}{r^{2n+3}}$$

$$-2(1-\nu) \sum_{n=0}^\infty b_{2n+1} \frac{P'_{2n+1}(0)}{r^{2n+3}} = -p(r),$$

$$1 \le r \le \gamma, \qquad (2.16)$$

where prime indicates the differentiation with respect to the argument.

By substituting (2.15) into (2.16), it reduces to

$$-\frac{2}{\pi} \int_1^\gamma tg(t)R(r,t)dt - \sum_{n=0}^\infty r^{-(2n+3)}\{P_{2n}(0)$$

$$\times (2n+1)^2 a_{2n} + \alpha b_{2n+1} P'_{2n+1}(0)\} = -p(r),$$

$$1 \le r \le \gamma, \qquad (2.17)$$

where

$$R(r,t) = \frac{1}{r^2 - t^2} E\left(\frac{r}{t}\right), \quad t > r,$$

$$= \frac{r}{t} \frac{1}{r^2 - t^2} E\left(\frac{t}{r}\right) - \frac{1}{rt} K\left(\frac{t}{r}\right), \quad r > t. \qquad (2.18)$$

K and E in (2.18) are complete elliptic integrals of the first and the second kind, respectively and $\alpha = 2(1-\nu)$.

The solution will be complete, if the conditions on the surface of the spherical cavity are satisfied.

3. Conditions on the surface of the spherical cavity.

Equation (2.17) gives one relation connecting unknown coefficients a_n and b_n. The stress components besides (2.8) and (2.9) which are needed for the present analysis are given by the following equations

$$\sigma_{\zeta\zeta} = \frac{\partial^2 \phi}{\partial \zeta^2} + \zeta \cos\vartheta \frac{\partial^2 \psi}{\partial \zeta^2} - 2(1-\nu)\cos\vartheta \frac{\partial \psi}{\partial \zeta}$$

$$+2\nu \frac{\sin\vartheta}{\zeta} \frac{\partial \psi}{\partial \vartheta}, \qquad (3.1)$$

$$\sigma_{\zeta\vartheta} = \frac{1}{\zeta} \frac{\partial^2 \phi}{\partial \zeta \partial \vartheta} - \frac{1}{\zeta^2} \frac{\partial \phi}{\partial \vartheta} + \cos\vartheta \frac{\partial^2 \psi}{\partial \zeta \partial \vartheta}$$

$$+(1-2\nu)\sin\vartheta \frac{\partial \psi}{\partial \zeta} - 2(1-\nu)\frac{\cos\vartheta}{\zeta} \frac{\partial \psi}{\partial \vartheta}. \qquad (3.2)$$

To satisfy boundary conditions on the spherical surface, it is needed to represent ϕ, ψ in (2.10)-(2.13) in terms of ζ, ϑ variables. To do so we utilize the following formula whose validity is shown in the Appendix 1. An expression useful for the present analysis is the following

$$\frac{P_n(\cos\theta)}{\rho^{n+1}} = \sum_{k=0}^\infty \binom{n+k}{k} \frac{P_{n+k}(\cos\vartheta)}{\zeta^{n+k+1}} \delta^k. \qquad (3.3)$$

Thus

$$\sum_{n=0}^\infty a_n \frac{P_n(\cos\theta)}{\rho^{n+1}} = \sum_{n=0}^\infty a_n \sum_{k=0}^\infty \binom{n+k}{k}$$

$$\times \frac{P_{n+k}(\cos\vartheta)}{\zeta^{n+k+1}} \delta^k = \sum_{j=0}^\infty \frac{P_j(\cos\vartheta)}{\zeta^{j+1}} A_j, \qquad (3.4)$$

where

$$A_j = \sum_{n=0}^j \frac{j!}{(j-n)!n!} a_n \delta^{j-n}. \qquad (3.5)$$

Also

$$\int_0^\infty \xi^{-1} A(\xi) J_0(\xi r) e^{-\xi z} d\xi = \int_1^\gamma tg(t)$$

$$\times \int_0^\infty \xi^{-1} J_0(\xi r) J_1(\xi t) e^{-\xi z} d\xi dt. \qquad (3.6)$$

If we make use of the formula in Whittaker and Watson[9,pp.395-396]

$$\int_{-\pi}^\pi \exp\{-\xi(z+ix\cos u+iy\sin u)\}du = 2\pi e^{-\xi z} J_0(\xi r),$$

to the inner integral of (3.6), it can be written as, if we are using the shortened notation

$$\beta = z + ix\cos u + iy\sin u,$$

then

$$\int_0^\infty \xi^{-1} J_0(\xi r) J_1(\xi t) e^{-\xi z} d\xi$$

$$= \frac{1}{2\pi} \int_{-\pi}^\pi \int_0^\infty \xi^{-1} J_1(\xi t) e^{-\xi\beta} d\xi du$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{t}{\beta + \sqrt{\beta^2 + t^2}} du$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\beta}{t} - \sqrt{1 + \frac{\beta^2}{t^2}} \right\} du$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\beta}{t} - \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n (-1)^n}{n!} \left(\frac{\beta}{t} \right)^{2n} \right\} du. \tag{3.7}$$

As $|\beta| < |t|$, the above series expansion is valid. Next,

$$\beta^{2n} = (-\delta + Z + ix \cos u + iy \sin u)^{2n} = (-\delta + \beta')^{2n}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} (-\delta)^{2n-k} (\beta')^k, \tag{3.8}$$

where (r, ϕ, Z) is the cylindrical coordinates system centered at $(r, z) = (0, -\delta)$. We have the following formula from Whittaker and Watson[9,p.392]

$$\int_{-\pi}^{\pi} (\beta')^n du = \int_{-\pi}^{\pi} (Z + ix \cos u + iy \sin u)^n du$$

$$= 2\pi \zeta^n P_n(\cos \vartheta). \tag{3.9}$$

If we utilize (3.8) and (3.9) in (3.7), it can be written as

$$\int_0^\infty \xi^{-1} J_0(\xi r) J_1(\xi t) e^{-\xi z} d\xi$$

$$= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n (-1)^n}{n!} \frac{1}{t^{2n}}$$

$$\times \sum_{k=0}^{2n} \binom{2n}{k} (-\delta)^{2n-k} \zeta^k P_n(\cos \vartheta)$$

$$+ \delta - \frac{\zeta}{t} P_1(\cos \vartheta). \tag{3.10}$$

Thus finally, from (3.4), (3.6) and (3.10), $\phi^{(1)}$ can be written in terms of spherical coordinates (ζ, ϑ) as

$$\phi^{(1)} = \sum_{k=0}^{\infty} P_k(\cos \vartheta) \left[\frac{A_k}{\zeta^{k+1}} + \Phi_k \zeta^k \right]$$

$$+ \delta \int_1^\gamma t g(t) dt - \zeta P_1(\cos \vartheta) \int_1^\gamma g(t) dt,$$

where

$$\Phi_k = -(\alpha - 1) \sum_{n=[(k+1)/2]}^{\infty} \frac{(-\frac{1}{2})_n (-1)^n}{n!}$$

$$\times \frac{(2n)!}{(2n-k)!k!} (-\delta)^{2n-k} \int_1^\gamma \frac{g(t)}{t^{2n-1}} dt, \tag{3.11}$$

and $[(k+1)/2]$ is the greatest integer $\leq (k+1)/2$.

It is also necessary to express $\psi^{(1)}$ in terms of spherical coordinates (ζ, ϑ). Now, as in (3.4)

$$\sum_{n=0}^{\infty} b_n \frac{P_n(\cos \theta)}{\rho^{n+1}} = \sum_{j=0}^{\infty} \frac{P_j(\cos \vartheta)}{\zeta^{j+1}} B_j, \tag{3.12}$$

where

$$B_j = \sum_{n=0}^{\infty} \frac{j!}{(j-n)!n!} b_n \delta^{j-n}.$$

We first express following integral in (ρ, θ) coordinates

$$\int_0^\infty A(\xi) J_0(\xi r) e^{-\xi z} d\xi = \int_1^\gamma t g(t) dt$$

$$\times \int_0^\infty J_1(\xi t) J_0(\xi r) e^{-\xi z} d\xi. \tag{3.13}$$

The inner integral on the right-hand side of (3.13) is

$$-\frac{1}{t} \int_0^\infty J_0(\xi r) e^{-\xi z} \frac{\partial}{\partial \xi} J_0(\xi t) d\xi = \frac{1}{t}$$

$$-\frac{1}{t} \int_0^\infty \{ r J_1(\xi r) + z J_0(\xi r) \} e^{-\xi z} J_0(\xi t) d\xi. \tag{3.14}$$

Using the equation in Erdélyi et al.[10] [0]

$$J_0(\xi t) = \frac{2}{\pi} \int_t^\infty \frac{\sin(\xi x) dx}{\sqrt{x^2 - t^2}},$$

equation (3.14) is equal to

$$\frac{1}{t} + \frac{1}{t} \frac{2}{\pi} \int_t^\infty \frac{dx}{\sqrt{x^2 - t^2}}$$

$$\times \Im \int_0^\infty \{ r J_1(\xi r) + z J_0(\xi r) \} e^{-\xi(z+ix)} d\xi$$

$$= \frac{1}{t} + \frac{1}{t} \frac{2}{\pi} \int_t^\infty \frac{1}{\sqrt{x^2 - t^2}} \Im \frac{-ix}{\sqrt{r^2 + (z+ix)^2}} dx$$

$$= \frac{1}{t} - \frac{1}{t} \frac{2}{\pi} \int_t^\infty \frac{1}{\sqrt{x^2 - t^2}} \Im \frac{dx}{\sqrt{1 - 2\frac{\rho}{x} i \cos \theta + (\frac{\rho}{x} i)^2}}$$

$$= \frac{1}{t} - \frac{1}{t} \sum_{n=0}^{\infty} \left(\frac{\rho}{t} \right)^{2n+1} (-1)^n P_{2n+1}(\cos \theta) \frac{(\frac{1}{2})_n}{n!}, \tag{3.15}$$

where we used the generating function

$$\frac{1}{\sqrt{1 - 2x\cos\theta + x^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)x^n.$$

To express (3.15) in the (ζ, ϑ) spherical coordinates, we use the following formula whose validity is shown in the Appendix 2.

$$\rho^{2n+1}P_{2n+1}(\cos\theta) = \sum_{k=0}^{2n+1} \frac{(2n+1)!}{k!(2n+1-k)!}$$

$$\times(-\delta)^{2n+1-k}\zeta^k P_k(\cos\vartheta). \qquad (3.16)$$

[0] If we use (3.16) in (3.15), then with (3.12), we get $\psi^{(1)}$ in spherical coordinates (ζ, ϑ)

$$\psi^{(1)} = \sum_{k=0}^{\infty} P_k(\cos\vartheta)\left(\frac{B_k}{\zeta^{k+1}} + \Psi_k\zeta^k\right) + \int_1^\gamma g(t)dt, \qquad (3.17)$$

where

$$\Psi_k = -\sum_{n=[k/2]}^{\infty} \frac{(-1)^n(\frac{1}{2})_n}{n!} \frac{(2n+1)!}{k!(2n+1-k)!}$$

$$\times(-\delta)^{2n+1-k}\int_1^\gamma \frac{g(t)}{t^{2n+1}}dt.$$

Then if we substitute these values of $\phi^{(1)}$ and $\psi^{(1)}$ given by (3.11) and (3.17) into (3.2), and simplify the results by using the properties of Legendre polynomials, from the condition that on the spherical surface $\zeta = \zeta_0$, the shear stress $\sigma_{\zeta\vartheta} = 0$, we obtain the following equation

$$-A_{k+1}\frac{k+3}{\zeta_0^{k+4}} + \frac{B_k}{\zeta_0^{k+2}}\frac{\alpha - (k+1)^2}{2k+1}$$

$$-\frac{B_{k+2}}{\zeta_0^{k+4}}\frac{(k+3)(2\alpha+k+2)}{2k+5}$$

$$+k\zeta_0^{k-1}\Phi_{k+1} + \zeta_0^{k-1}\Psi_k\frac{k(k+1-2\alpha)}{2k+1}$$

$$-\zeta_0^{k+1}\Psi_{k+2}\frac{k+2+\alpha-(k+2)(k+3)}{2k+5} = 0. \qquad (3.18)$$

Likewise, from the condition $\sigma_{\zeta\zeta}(\zeta_0, \vartheta) = 0$, we get following two equations,

$$-A_0\frac{2}{\zeta_0^3} - \frac{B_1}{\zeta_0^3}\frac{2(2\alpha+1)}{3} - \frac{1}{3}\Psi_1(\alpha - 4) = 0, \qquad (3.19a)$$

$$-A_{k+1}\frac{(k+2)(k+3)}{\zeta_0^{k+4}}$$

$$+\frac{B_k}{\zeta_0^{k+2}}\frac{(k+1)\{2-\alpha-(k+1)(k+4)\}}{2k+1}$$

$$-\frac{B_{k+2}}{\zeta_0^{k+4}}\frac{(k+2)(k+3)(2\alpha+k+2)}{2k+5}$$

$$-k(k+1)\zeta_0^{k-1}\Phi_{k+1} - \zeta_0^{k-1}\Psi_k\frac{k(k+1)(k+1-2\alpha)}{2k+1}$$

$$-\zeta_0^{k+1}\Psi_{k+2}\frac{(k+2)\{(k+2)(k-1)+\alpha-2\}}{2k+5} = 0. \qquad (3.19b)$$

Thus if we multiply (3.18) by $k+2$ and subtract (3.19b) from the resulting equation we find

$$B_k = -\frac{\zeta_0^{k+2}(2k+1)}{\alpha(2k+3)+2k(k+1)}\left[k(2k+3)\zeta_0^{k-1}\Phi_{k+1}\right.$$

$$\left.+\zeta_0^{k-1}\Psi_k\frac{(2k+3)k(k+1-2\alpha)}{2k+1}+\zeta_0^{k+1}\Psi_{k+2}k(k+2)\right]. \qquad (3.20)$$

If we solve (3.20) for b_i using the theorem which is in the Appendix 3 and the relation,

$$\Phi_{k+1} = \frac{\alpha-1}{k+1}\Psi_k,$$

we find

$$\frac{b_i}{\delta^i} = \sum_{k=0}^{i}\frac{i!(-1)^{i-k}}{(i-k)!k!}\frac{1}{\delta^k}[N_1(k)\Psi_k + N_2(k)\Psi_{k+2}], \qquad (3.21)$$

where

$$N_1(k) = -\frac{\zeta_0^{2k+1}(2k+3)k}{\alpha(2k+3)+2k(k+1)}\left(\frac{-\alpha+k^2}{k+1}\right),$$

$$N_2(k) = -\frac{\zeta_0^{2k+3}(2k+1)k(k+2)}{\alpha(2k+3)+2k(k+1)}.$$

Equation (3.21) can be written as

$$b_i = -\sum_{k=0}^{i}\frac{i!(-1)^{i-k}}{(i-k)!k!}\frac{\delta^i}{\delta^k}\int_1^\gamma\frac{g(t)}{t}$$

$$\times\sum_{n=0}^{1}[N_{n+1}(k)f_{k+2n}\left(\frac{1}{2},\frac{\delta}{t}\right)dt, \qquad (3.22)$$

where

$$f_k(c,x) = \frac{(-\delta)^{1-k}}{k!} \sum_{\ell=[k/2]}^{\infty} \frac{(-1)^{\ell}(c)_{\ell}\,(2\ell+1)!x^{2\ell}}{(2\ell+1-k)!\ell!}$$

with $(c)_{\ell} = c(c+1)(c+2)\cdots(c+\ell-1)$.

Now from (3.19a,b) we have

$$A_{k+1} = M_1(k)\Psi_k + M_2(k)\Psi_{k+2} + M_3(k)\Psi_{k+4},$$

$$A_0 = \tilde{A}_1\Psi_1 + \tilde{A}_2\Psi_3,$$

where [0]

$$M_1(k) = -\frac{\zeta_0^{2k+3}k(-\alpha+k^2)}{(k+2)(k+3)(2k+1)}$$
$$\times\left[\frac{(2k+3)\tilde{g}(k)}{h(k)} + 1\right],$$

$$M_2(k) = \frac{\zeta_0^{2k+5}}{k+3}\left[-\frac{\tilde{g}(k)}{h(k)}k(k+1) + \frac{\tilde{g}(k-2)}{2k+5}\right.$$

$$\left.+\frac{(k+2)(2k+7)(2\alpha+k+2)\{-\alpha+(k+2)^2\}}{(2k+5)h(k+2)}\right],$$

$$M_3(k) = \frac{\zeta_0^{2k+7}(k+2)(k+4)(2\alpha+k+2)}{h(k+2)},$$

$$\tilde{A}_1 = -\frac{\zeta_0^3}{6}\left\{\alpha - 4 - \frac{5(2\alpha+1)(-\alpha+1)}{h(1)}\right\},$$

$$\tilde{A}_2 = \frac{3(2\alpha+1)\zeta_0^5}{h(1)},$$

with

$$\tilde{g}(k) = 2 - \alpha - (k+1)(k+4),$$
$$h(k) = \alpha(2k+3) + 2k(k+1).$$

In a similar way $a_i's$ can be found from these equations as follows:

$$a_{2i} = (\tilde{A}_1\Psi_1 + \tilde{A}_2\Psi_3)\delta^{2i}$$
$$+\sum_{k=0}^{2i-1}\frac{(2i)!(-\delta)^{2i-k-1}}{(2i-1-k)!(k+1)!}\tilde{F}(k),$$

where

$$\tilde{F}(k) = -\int_1^{\gamma}\frac{g(t)}{t}\left[M_1(k)f_k\left(\frac{1}{2},\frac{\delta}{t}\right) + M_2(k)\right.$$
$$\left.\times f_{k+2}\left(\frac{1}{2},\frac{\delta}{t}\right) + M_3(k)f_{k+4}\left(\frac{1}{2},\frac{\delta}{t}\right)\right]dt.$$

Thus

$$\sum_{i=0}^{\infty} a_{2i}\frac{(2i+1)^2 P_{2i}(0)}{r^{2i+3}} = \frac{a_0}{r^3} + \sum_{i=1}^{\infty}(\tilde{A}_1\Psi_1 + \tilde{A}_2\Psi_3)$$

$$\times\delta^{2i}\frac{(2i+1)^2(-1)^i(\frac{1}{2})_i}{r^{2i+3}i!}$$

$$+\sum_{i=1}^{\infty}\frac{(2i+1)^2(-1)^i(\frac{1}{2})_i}{r^{2i+3}i!}$$

$$\times\sum_{k=0}^{2i-1}\frac{(2i)!(-\delta)^{2i-k-1}}{(2i-1-k)!(k+1)!}\tilde{F}(k).$$

If we interchange the order of summation in the last term of the above equation, and use

$$\Psi_1 = -\int_1^{\gamma}\frac{g(t)}{t}f_1\left(\frac{1}{2},\frac{\delta}{t}\right)dt,$$

$$\Psi_3 = -\int_1^{\gamma}\frac{g(t)}{t}f_3\left(\frac{1}{2},\frac{\delta}{t}\right)dt,$$

we finally obtain

$$\sum_{i=0}^{\infty} a_{2i}\frac{(2i+1)^2 P_{2i}(0)}{r^{2i+3}} = -\int_1^{\gamma} tg(t)T_1(r,t)dt,$$

where

$$T_1(r,t) = \frac{1}{r^3 t^2}$$

$$\times\left[\sum_{k=0}^{\infty}\sum_{m=0}^{2} M_{m+1}(k)f_{k+2m}\left(\frac{1}{2},\frac{\delta}{t}\right)h_k\left(\frac{\delta}{r}\right)\right.$$

$$\left.+\left\{\tilde{A}_1 f_1\left(\frac{1}{2},\frac{\delta}{t}\right) + \tilde{A}_2 f_3\left(\frac{1}{2},\frac{\delta}{t}\right)\right\}\left\{1 + j\left(\frac{\delta}{r}\right)\right\}\right],$$

with

$$h_k\left(\frac{\delta}{r}\right) = \frac{(-\delta)^{-k-1}}{(k+1)!}$$

$$\times\sum_{i=[k/2+1]}^{\infty}\frac{(2i+1)^2(-1)^i(\frac{1}{2})_i(2i)!}{(2i-1-k)!i!}\left(\frac{\delta}{r}\right)^{2i},$$

$$j\left(\frac{\delta}{r}\right) = \sum_{i=1}^{\infty}\frac{(2i+1)^2(-1)^i(\frac{1}{2})_i}{i!}\left(\frac{\delta}{r}\right)^{2i}.$$

Also,

$$\sum_{i=0}^{\infty} b_{2i+1}\frac{P'_{2i+1}(0)}{r^{2i+3}} = -\sum_{i=0}^{\infty}\frac{(-1)^i(\frac{3}{2})_i}{i!r^{2i+3}}$$

$$\times\sum_{k=0}^{2i+1}\frac{(2i+1)!(-\delta)^{2i-k+1}}{(2i+1-k)!k!}$$

$$\times \int_1^\gamma \frac{g(t)}{t} \sum_{n=0}^1 N_{n+1}(k) f_{k+2n}\left(\frac{1}{2}, \frac{\delta}{t}\right) dt.$$

If we change the order of summation in the above equation, we get

$$\sum_{i=0}^\infty b_{2i+1} \frac{P'_{2i+1}(0)}{r^{2i+3}} = -\int_1^\gamma tg(t) T_2(r,t) dt,$$

where

$$T_2(r,t) = \frac{1}{t^2 r^3} \sum_{k=0}^\infty \sum_{n=0}^1 N_{n+1}(k) f_k\left(\frac{3}{2}, \frac{\delta}{r}\right)$$

$$\times f_{k+2n}\left(\frac{1}{2}, \frac{\delta}{t}\right).$$

Thus finally the singular integral equation (2.17) becomes

$$\frac{2}{\pi} \int_1^\gamma tg(t)\{R(r,t) + S(r,t)\} dt = p(r),$$

$$1 \le r \le \gamma, \qquad (3.23)$$

where

$$S(r,t) = -\frac{\pi}{2}\{T_1(r,t) + \alpha T_2(r,t)\}.$$

When $\delta = 0$, we briefly show that this equation completely agrees with what Atsumi and Shindo obtained. Using

$$\lim_{\delta \to 0} h_{2k-1}\left(\frac{\delta}{r}\right) = \frac{(-1)^k (\frac{1}{2})_k (2k+1)^2}{k!} \frac{1}{r^{2k}},$$

$$\lim_{\delta \to 0} f_{2k-1}\left(\frac{1}{2}, \frac{\delta}{t}\right) = \frac{(-1)^{k-1}(\frac{1}{2})_{k-1}}{(k-1)!} \frac{1}{t^{2k-2}},$$

$$A = \lim_{\delta \to 0} \sum_{k=0}^\infty \sum_{m=0}^2 M_{m+1}(k) f_{k+2m}\left(\frac{1}{2}, \frac{\delta}{t}\right) h_k\left(\frac{\delta}{r}\right)$$

$$= \sum_{k=1}^\infty \frac{(-1)^k (\frac{1}{2})_k (2k+1)^2}{k! r^{2k}}$$

$$\times \frac{1}{(2k+1)(2k+2)} \left[\frac{2k-1}{4k-1}\left\{\frac{(4k+1)G(k)}{H(k)} + 1\right\}\right.$$

$$\times F(k) \frac{(-1)^{k-1}(\frac{1}{2})_{k-1}}{(k-1)!} \frac{1}{t^{2k-2}}$$

$$+ (2k+1)\left\{-\frac{G(k)}{H(k)}(2k-1)2k\right.$$

$$\left.- \frac{(2k+1)(4k+5)(2\alpha+2k+1)}{H(k+1)(4k+3)} F(k+1)\right.$$

$$\left. + \frac{G(k-1)}{4k+3}\right\} \frac{(-1)^k (\frac{1}{2})_k}{k!} \frac{1}{t^{2k}}$$

$$+ \frac{(2k+1)^2(2k+2)(2k+3)(2\alpha+2k+1)}{H(k+1)}$$

$$\left.\times \frac{(-1)^{k+1}(\frac{1}{2})_{k+1}}{(k+1)!} \frac{1}{t^{2k+2}}\right],$$

where [0]

$$H(k) = \alpha(4k+1) + 4k(2k-1),$$
$$G(k) = 2 - \alpha - 2k(2k+3),$$
$$F(k) = -\alpha + (2k-1)^2.$$

$$B = \lim_{\delta \to 0} \sum_{k=0}^\infty \sum_{n=0}^1 N_{n+1}(k) f_k\left(\frac{3}{2}, \frac{\delta}{r}\right) f_{k+2n}\left(\frac{1}{2}, \frac{\delta}{t}\right)$$

$$= \sum_{k=1}^\infty \frac{(-1)^k (\frac{3}{2})_k}{k! r^{2k}} \left[-\frac{(2k+1)(4k+5)}{2(k+1)}\right.$$

$$\times \frac{F(k+1)}{H(k+1)} \frac{(-1)^k (\frac{1}{2})_k}{k!} \frac{1}{t^{2k}} - \frac{(4k+3)(2k+1)(2k+3)}{H(k+1)}$$

$$\left.\times \frac{(-1)^{k+1}(\frac{1}{2})_{k+1}}{(k+1)!} \frac{1}{t^{2k+2}}\right] + \frac{1}{2H(1)}\left\{-5F(1) + \frac{9}{t^2}\right\}.$$

Thus

$$A + \alpha B = \sum_{k=1}^\infty \frac{(-1)^k (\frac{1}{2})_k (2k+1)}{k! r^{2k}}$$

$$\times \left[\frac{(-1)^k (2k-1)(2k+1)}{H(k)k(k+1)}\left\{k(k+1)F(k)\right.\right.$$

$$\left.- \frac{1}{t^2}\frac{1}{4(4k+3)}\{(2k-1)(4k+3)2kG(k)\right.$$

$$\left.\left.- H(k)G(k-1)\}\right\} \frac{(\frac{1}{2})_{k-1}}{(k-1)!} \frac{1}{t^{2k-2}}\right.$$

$$+ (-1)^k \frac{F(k+1)(2k+1)}{H(k+1)(2k+2)}\left\{(2k+3)(2k+2)\frac{1}{t^2}\right.$$

$$\left.\left.+ F(k+1)\frac{4k+5}{4k+3}\right\} \frac{(\frac{1}{2})_k}{k!} \frac{1}{t^{2k}}\right] + \frac{\alpha}{2H(1)}\left\{\frac{9}{t^2} - 5F(1)\right\}$$

$$= \sum_{k=1}^\infty \frac{2(2k+1)^2}{H(k)(k+1)}\left\{k(k+1)F(k) - \frac{1}{t^2}\frac{1}{4(4k+3)}\right.$$

$$\left.\times \{(2k-1)(4k+3)2kG(k) - H(k)G(k-1)\}\right\}$$

$$\times \frac{1}{(rt)^{2k-2}r^2}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2$$

$$+ \sum_{k=2}^\infty \frac{F(k)}{H(k)}\left\{(2k+1)(2k-1)\frac{1}{t^2} + F(k)\frac{4k+1}{4k-1}\right\}$$

$$\times 2k\left(\frac{(\frac{1}{2})_k}{k!}\right)^2 \frac{1}{(rt)^{2k-2}} + \frac{\alpha}{2H(1)}\left\{-5F(1)+\frac{9}{t^2}\right\}.$$

Using

$$\tilde{A}_1 f_1\left(\frac{1}{2},0\right) + \tilde{A}_2 f_3\left(\frac{1}{2},0\right) = \tilde{A}_1 - \tilde{A}_2\frac{1}{2t^2},$$

we finally obtain

$$-\frac{2}{\pi}S(r,t) = \frac{1+\nu}{3}\frac{1}{r^3t^2} + \sum_{k=1}^{\infty}\frac{2}{H(k)}\left[\frac{2(2k+1)^2}{(k+1)r^2}\right.$$

$$\times\left\{k(k+1)F(k) - \frac{1}{t^2}\frac{1}{4(4k+3)}\{(2k-1)(4k+3)\right.$$

$$\times 2kG(k) - H(k)G(k-1)\} + kF(k)\left\{(2k+2)\right.$$

$$\times(2k-1)\frac{1}{t^2} + F(k)\frac{4k+1}{4k-1}\bigg\}\bigg]$$

$$\times\left(\frac{(2k-1)!!}{(2k)!!}\right)^2\frac{1}{(rt)^{2k}r}.$$

The above equation completely agrees with the result by Atsumi and Shindo.

A quantity of physical interest is the stress intensity factor which is given as

$$K = \lim_{r\to\gamma^+}\sqrt{2(r-\gamma)}\sigma_{zz}(r,0).$$

We choose $p(r) = p_0$ and put $\gamma - 1 = a$. We let

$$r = \frac{a}{2}(s+1)+1, \quad t = \frac{a}{2}(\tau+1)+1, \quad (3.24)$$

and in order to facilitate numerical analysis, assume $g(t)$ to have the following form:

$$g(t) = p_0(t-1)^{\frac{1}{2}}(\gamma-t)^{-\frac{1}{2}}\tilde{G}(t). \quad (3.25)$$

With the aid of (3.24), $g(\tau)$ can be rewritten as

$$g(\tau) = p_0\tilde{G}(\tau)\left(\frac{1+\tau}{1-\tau}\right)^{\frac{1}{2}}. \quad (3.26)$$

The stress intensity factors K can therefore be expressed in terms of $\tilde{G}(t)$ as

$$K/p_0 = \sqrt{2a}\tilde{G}(\gamma), \quad (3.27)$$

or in terms of the quantity actually calculated

$$K/p_0\sqrt{a} = \sqrt{2}\tilde{G}(\gamma). \quad (3.28)$$

4.Numerical analysis.

In order to obtain numerical solution of (3.23), substitutions are made by the application of

(3.24) and (3.26) to obtain the following expression:

$$\frac{a}{\pi}\int_{-1}^{1}\left(\frac{1+\tau}{1-\tau}\right)^{\frac{1}{2}}\tilde{G}(\tau)\left[\frac{a}{2}(\tau+1)+1\right][R(s,\tau)$$

$$+S(s,\tau)]d\tau = 1, \quad -1 < s < 1. \quad (4.1)$$

The numerical solution technique is based on the collocation scheme for the solution of singular integral equations given by Erdogan, Gupta, and Cook [11]. [0] This amounts to applying a Gaussian quadrature formula for approximating the integral of a function $f(\tau)$ with weight function $[(1+\tau)/(1-\tau)]^{\frac{1}{2}}$ on the interval [-1,1]. Thus, letting n be the number of quadrature points,

$$\int_{-1}^{1}\left(\frac{1+\tau}{1-\tau}\right)^{\frac{1}{2}}f(\tau)d\tau \doteq \frac{2\pi}{2n+1}\sum_{k=1}^{n}(1+\tau_k)f(\tau_k),$$

$$(4.2)$$

where

$$\tau_k = \cos\left(\frac{2k-1}{2n+1}\right)\pi, \quad k = 1,\ldots n. \quad (4.3)$$

The solution of the integral equation is obtained by choosing the collocation points:

$$s_i = \cos\left(\frac{2i\pi}{2n+1}\right), \quad i = 1,\ldots,n, \quad (4.4)$$

and solving the matrix system for $G^*(\tau_k)$:

$$\sum_{k=1}^{n}[R(s_j,\tau_k)+S(s_j,\tau_k)]G^*(\tau_k) = \frac{2n+1}{2a},$$

$$j = 1,\ldots,n, \quad (4.5)$$

where

$$\tilde{G}(\tau_k) = \frac{G^*(\tau_k)}{(1+\tau_k)[\frac{a}{2}(\tau_k+1)+1]}. \quad (4.6)$$

5.Numerical results and consideration.

Numerical calculations have been carried out for $\nu = 0.3$. The values of normalized stress intensity factor $K/p_0\sqrt{a}$ versus a are shown in Fig.1-3 for various values of δ.

Fig.1 shows the variation of $K/p_0\sqrt{a}$ with respect to a when $\delta = 0$.

This figure shows that as a increases, S.I.F. decreases steadily.

Fig.2 and 3 deal with the cases when $\delta = 0.3$ and $\delta = 0.5$, respectively. Here we omit units. We can see that the trend is similar. Theoretically the infinite series involved converges when $\delta < 1$ by comparison test. However, because of the overflow, computations could not be accomplished beyond the values $\delta > 0.5$. And we found that the variation of SIF is very small with respect to the variation of δ.

0

Fig.1 Variation of s.i.f. when delta=0

Fig.2. variation of s.i.f. when delta=.1

Fig.3 Variation of s.i.f. when delta=.5
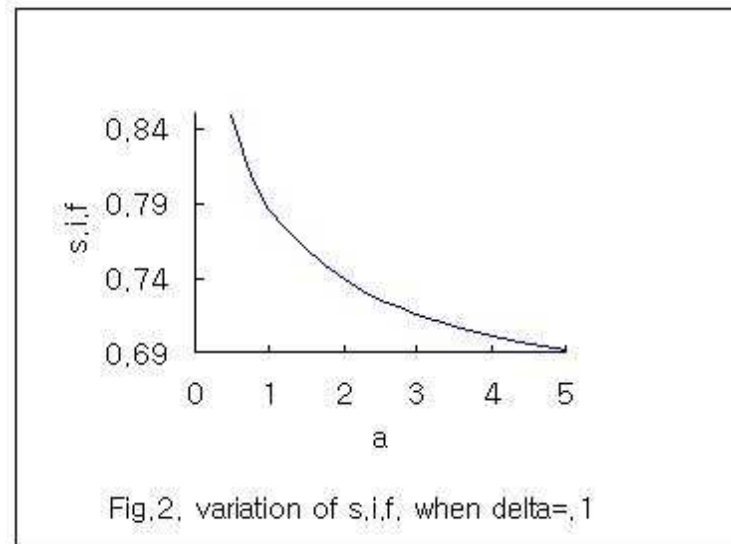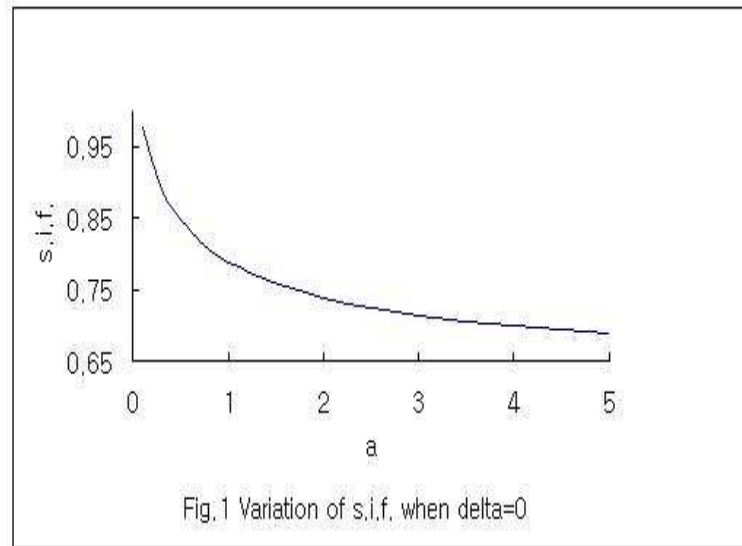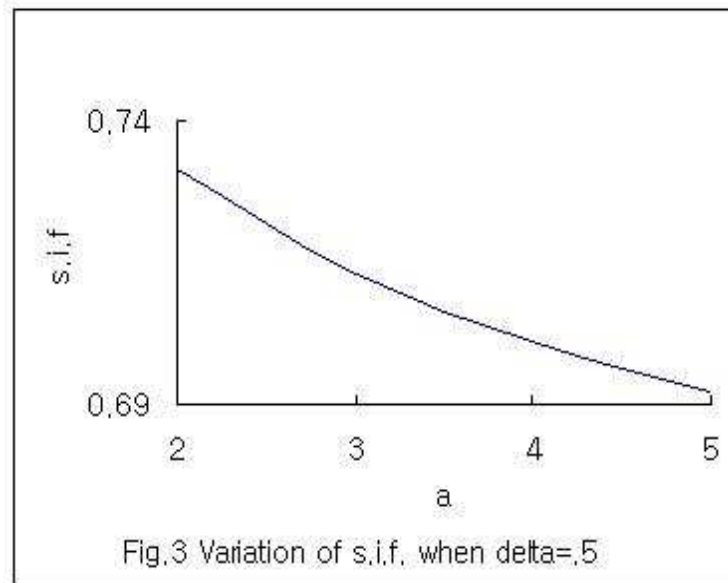
6.Penny-shaped crack.

In this section we are concerned with a penny-shaped crack in a convex lens shaped elastic material. The problem of determining the distribution of stress in an elastic sphere containing a penny-shaped crack or the mixed boundary value problems concerning a spherical boundary has been investigated by several researchers. Srivastava and Dwivedi [12] considered the problem of a penny-shaped crack in an elastic sphere, whereas Dhaliwal et al.[13] solved the problem of a penny-shaped crack in a sphere embedded in an infinite medium. On the other hand, Srivastav and Narain [14] investigated the mixed boundary value problem of torsion of a hemisphere.

7.Formulation of problem and reduction to a Fredholm integral equation of the second kind. [0] We employ cylindrical coordinates (r, ϕ, z) with the plane $z = 0$ coinciding the plane of the penny shaped crack. The center of the crack is located at (r,z)=(0,0). As before, spherical coordinates (ρ, θ, ϕ) are connected with the cylindrical coordinates by

$$z = \rho \cos \theta, \qquad r = \rho \sin \theta.$$

Spherical coordinates (ζ, ϑ, ϕ) whose origin is at $z = -\delta, \quad r = 0$, and is the center of the upper spherical surface of the convex elastic body, is

also used. The elastic body is symmetrical with respect to the plane $z = 0$.

The crack occupies the region $z = 0, \ 0 \le r \le 1$. The radius of the spherical portion is $\zeta_0 = \sqrt{\gamma^2 + \delta^2}$. The boundary conditions are:

On the plane $z = 0$, we want the continuity of the shear stress, and the normal displacement:

$$u_z(r, 0^+) - u_z(r, 0^-) = 0, \qquad 1 \le r \le \gamma, \quad (7.1)$$

$$\sigma_{rz}(r, 0^+) - \sigma_{rz}(r, 0^-) = 0, \qquad 0 \le r \le \gamma. \quad (7.2)$$

And the crack is subjected to a known pressure $p(r)$, i.e.,

$$\sigma_{zz}(r, 0^+) = -p(r), \qquad 0 \le r \le 1. \qquad (7.3)$$

On the surface of the spherical portion, stresses are zero:

$$\sigma_{\zeta\zeta}(\zeta_0, \vartheta) = 0, \qquad\qquad (7.4)$$
$$\sigma_{\zeta\vartheta}(\zeta_0, \vartheta) = 0. \qquad\qquad (7.5)$$

We can make use of (2.6)-(2.9) also, for the present case. The functions $\phi^{(1)}$ and $\phi^{(2)}$ for the regions $z > 0$ and $z < 0$, respectively, are chosen as follows:

$$\phi^{(1)}(r, z) = (2\nu - 1) \int_0^\infty \xi^{-1} A(\xi) J_0(\xi r) e^{-\xi z} d\xi$$

$$+ \sum_{n=0}^{\infty} a_n \rho^n P_n(\cos\theta), \qquad (7.6)$$

$$\phi^{(2)}(r,z) = (2\nu - 1) \int_0^{\infty} \xi^{-1} A(\xi) J_0(\xi r) e^{\xi z} d\xi$$

$$- \sum_{n=0}^{\infty} a_n (-1)^n \rho^n P_n(\cos\theta). \qquad (7.7)$$

Here the superscripts (1) and (2) are taken for the region $z > 0$ and $z < 0$, respectively. The functions $\psi^{(1)}$ and $\psi^{(2)}$ are chosen as follows:

$$\psi^{(1)}(r,z) = \int_0^{\infty} A(\xi) J_0(\xi r) e^{-\xi z} d\xi$$

$$+ \sum_{n=0}^{\infty} b_n \rho^n P_n(\cos\theta), \qquad (7.8)$$

$$\psi^{(2)}(r,z) = - \int_0^{\infty} A(\xi) J_0(\xi r) e^{\xi z} d\xi$$

$$+ \sum_{n=0}^{\infty} b_n (-1)^n \rho^n P_n(\cos\theta). \qquad (7.9)$$

Then we can immediately satisfy condition (7.2) by these choice of functions (7.6)-(7.9).

Now the condition (7.1) requires

$$\int_0^{\infty} A(\xi) J_0(\xi r) d\xi = 0, \quad r > 1. \qquad (7.10)$$

Equation (7.10) is automatically satisfied by setting

$$A(\xi) = \int_0^1 g(t) \sin(\xi t) dt, \ g(0) = 0. \qquad (7.11)$$

Then from the boundary condition (7.3), we obtain

$$\int_0^{\infty} \xi A(\xi) J_0(\xi r) d\xi - \sum_{n=0}^{\infty} a_{2n} (2n)^2 P_{2n}(0) r^{2n-2}$$

$$-2(1-\nu) \sum_{n=0}^{\infty} b_{2n+1} P'_{2n+1}(0) r^{2n} = -p(r),$$

$$0 \leq r \leq 1, \qquad (7.12)$$

where the prime indicates the differentiation with respect to the argument. [0]

By substituting (7.11) into (7.12), it reduces to

$$g(t) - \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \{ 2nt^{2n-1} a_{2n}$$

$$+\alpha b_{2n+1} t^{2n+1} \} = h(t), \quad 0 \leq r \leq 1, \qquad (7.13)$$

where

$$h(t) = -\frac{2}{\pi} \int_0^t \frac{rp(r)}{\sqrt{t^2 - r^2}} dr.$$

The solution will be complete, if the conditions on the surface of the spherical portion are satisfied.

8. Conditions on the surface of the sphere.

Equation (7.13) gives one relation connecting unknown coefficients a_n and b_n. The stress components besides (2.8) and (2.9) which are needed for the present analysis are given by (3.1) and (3.2). To satisfy boundary conditions on the spherical surface, it is needed to represent ϕ, ψ in (7.6)-(7.9) in terms of ζ, ϑ variables. To do so we utilize the following formula in the Appendix 2. An expression useful for the present analysis is the following

$$P_n(\cos\theta)\rho^n = \sum_{k=0}^{n} \binom{n}{k} P_k(\cos\vartheta) \zeta^k (-\delta)^{n-k}. \qquad (8.1)$$

Thus

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta)\rho^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \binom{n}{k} P_k(\cos\vartheta)$$

$$\times \zeta^k (-\delta)^{n-k} = \sum_{j=0}^{\infty} P_j(\cos\vartheta) \zeta^j A_j, \qquad (8.2)$$

where

$$A_j = \sum_{n=j}^{\infty} \frac{n!}{(n-j)!j!} a_n (-\delta)^{n-j}. \qquad (8.3)$$

Also

$$-\frac{\partial}{\partial t} \int_0^{\infty} \xi^{-1} A(\xi) J_0(\xi r) e^{-\xi z} d\xi$$

$$= - \int_0^1 g(t) \int_0^{\infty} J_0(\xi r) \cos(\xi t) e^{-\xi z} d\xi dt. \qquad (8.4)$$

The inner integral of the above equation is

$$\int_0^\infty J_0(\xi r)\cos(\xi t)e^{-\xi z}d\xi$$

$$= \Re \int_0^\infty J_0(\xi r)e^{-\xi(z+it)}d\xi$$

$$= \Re \frac{1}{\sqrt{r^2 + (z+it)^2}}$$

$$= \Re \frac{1}{\sqrt{r^2 + z^2 + 2zit - t^2}}$$

$$= \Re \frac{1}{\rho\sqrt{1 - 2\cos\theta(\frac{-it}{\rho}) + (\frac{-it}{\rho})^2}}$$

$$= \frac{1}{\rho}\sum_{n=0}^\infty \left(\frac{t}{\rho}\right)^{2n}(-1)^n P_{2n}(\cos\theta). \qquad (8.5)$$

Thus finally, from (8.2), and using the formula in Appendix, $\phi^{(1)}$ can be written in terms of spherical coordinates (ζ, ϑ) as

$$\phi^{(1)} = \sum_{k=0}^\infty P_k(\cos\vartheta)\left[\frac{\Phi_k}{\zeta^{k+1}} + A_k\zeta^k\right], \qquad (8.6)$$

where

$$\Phi_k = -(\alpha - 1)\sum_{n=0}^{[k/2]} \frac{k!}{(2n)!(k-2n)!}\delta^{k-2n}$$

$$\times (-1)^n \int_0^1 \frac{g(t)t^{2n+1}}{2n+1}dt. \qquad (8.7)$$

It is also necessary to express $\psi^{(1)}$ in terms of spherical coordinates (ζ, ϑ). Now, as in (8.2)

$$\sum_{n=0}^\infty b_n P_n(\cos\theta)\rho^n = \sum_{j=0}^\infty P_j(\cos\vartheta)\zeta^j B_j, \qquad (8.8)$$

where

$$B_j = \sum_{n=j}^\infty \frac{n!}{j!(n-j)!}b_n(-\delta)^{n-j}. \qquad (8.9)$$

[0] We first express following integral in (ρ, θ) coordinates

$$\int_0^\infty A(\xi)J_0(\xi r)e^{-\xi z}d\xi = \int_0^1 g(t)dt$$

$$\times \int_0^\infty \sin(\xi t)J_0(\xi r)e^{-\xi z}d\xi. \qquad (8.10)$$

The inner integral on the right-hand side of (8.10) is

$$-\Im \int_0^\infty J_0(\xi r)e^{-\xi(z+it)}d\xi$$

$$= \sum_{n=0}^\infty \frac{P_{2n+1}(\cos\theta)}{\rho^{2n+2}}(-1)^n t^{2n+1}. \qquad (8.11)$$

To express (8.11) in the (ζ, ϑ) spherical coordinates, we use the following formula in the Appendix 1.

$$\frac{P_{2n+1}(\cos\theta)}{\rho^{2n+1}} = \sum_{k=0}^\infty \frac{(2n+1+k)!}{k!(2n+1)!}$$

$$\times \delta^k \frac{P_{2n+1+k}(\cos\vartheta)}{\zeta^{2n+2+k}}. \qquad (8.12)$$

If we use (8.8) and (8.12), we get $\psi^{(1)}$ in spherical coordinates (ζ, ϑ) as

$$\psi^{(1)} = \sum_{k=0}^\infty P_k(\cos\vartheta)\left(\frac{\Psi_k}{\zeta^{k+1}} + B_k\zeta^k\right) + B_0, \qquad (8.13)$$

where

$$\Psi_k = \sum_{n=0}^{[(k-1)/2]} \frac{k!}{(2n+1)!(k-2n-1)!}\delta^{k-2n-1}$$

$$\times (-1)^n \int_0^1 g(t)t^{2n+1}dt. \qquad (8.14)$$

Then if we substitute these values of $\phi^{(1)}$ and $\psi^{(1)}$ given by (8.6) and (8.13) into (3.2), and simplify the results by using the properties of Legendre polynomials, from the condition that on the spherical surface $\zeta = \zeta_0$, $\sigma_{\zeta\vartheta} = 0$, we obtain the following equation

$$A_{k+1}k\zeta_0^{k-1} - B_{k+2}\zeta_0^{k+1}\frac{\alpha - (k+2)^2}{2k+5}$$

$$- B_k\zeta_0^{k-1}\frac{k(2\alpha - k - 1)}{2k+1} - \frac{k+3}{\zeta_0^{k+4}}\Phi_{k+1}$$

$$- \frac{\Psi_{k+2}}{\zeta_0^{k+4}}\frac{(k+3)(k+2+2\alpha)}{2k+5} + \frac{\Psi_k}{\zeta_0^{k+2}}\frac{\alpha - (k+1)^2}{2k+1}$$

$$= 0. \qquad (8.15)$$

Likewise, from the condition $\sigma_{\zeta\zeta}(\zeta_0, \vartheta) = 0$, we get the following equation,

$$-A_{k+1}k(k+1)\zeta_0^{k-1}$$

$$+B_{k+2}\zeta_0^{k+1}\frac{(k+2)\{2-\alpha-(k-1)(k+2)\}}{2k+5}$$

$$+B_k\zeta_0^{k-1}\frac{k(k+1)(2\alpha-k-1)}{(2k+1)}$$

$$-\frac{(k+2)(k+3)}{\zeta_0^{k+4}}\Phi_{k+1}$$

$$-\Psi_{k+2}\frac{(k+2)(k+3)(k+2+2\alpha)}{\zeta_0^{k+4}(2k+5)}$$

$$-\Psi_k\frac{(k+1)\{(k+4)(k+1)+\alpha-2\}}{\zeta_0^{k+2}(2k+1)}$$

$$= 0. \qquad (8.16)$$

Thus if we multiply (8.15) by $k+1$ and add (8.16), from the resulting equation we find

$$B_{k+2}=\frac{(2k+5)}{\zeta_0^{k+1}\{-\alpha(2k+3)+2(k+2)(k+3)\}}$$

$$\times\left[\frac{(k+3)(2k+3)}{\zeta_0^{k+4}}\Phi_{k+1}\right.$$

$$+\Psi_{k+2}\frac{(2k+3)(k+3)(k+2+2\alpha)}{\zeta_0^{k+4}(2k+5)}$$

$$\left.+\frac{\Psi_k}{\zeta_0^{k+2}}(k+1)(k+3)\right]. \qquad (8.17)$$

[0] Using the relation

$$\Phi_{k+1}=-\frac{\alpha-1}{k+2}\Psi_{k+2},$$

B_{k+2} is

$$B_{k+2}=\frac{(2k+5)}{\zeta_0^{k+1}\{-\alpha(2k+3)+2(k+2)(k+3)\}}$$

$$\times\left[\Psi_{k+2}\frac{(2k+3)(k+3)((k+3)^2-\alpha)}{\zeta_0^{k+4}(2k+5)(k+2)}\right.$$

$$\left.+\frac{\Psi_k}{\zeta_0^{k+2}}(k+1)(k+3)\right]. \qquad (8.18)$$

From (8.15) we have

$$A_{k+3}=\frac{\Psi_k}{\zeta_0^{2k+3}}L(k)+\frac{\Psi_{k+2}}{\zeta_0^{2k+5}}M(k)+\frac{\Psi_{k+4}}{\zeta_0^{2k+7}}N(k),$$

where

$$L(k)=(k+1)(k+3)\frac{(2\alpha-k-3)}{\mathcal{H}(k)},$$

$$M(k)=\frac{1}{k+2}\left[(k+3)\frac{(2k+3)(2\alpha-k-3)I(k)}{(2k+5)\mathcal{H}(k)}\right.$$

$$+(k+4)(k+5)\frac{G(k)}{\mathcal{H}(k+2)}+\frac{G(k+2)}{2k+5}\right],$$

$$N(k)=\left[\frac{G(k)}{\mathcal{H}(k+2)}(2k+7)-1\right]$$

$$\times\frac{(k+5)I(k+2)}{(2k+9)(k+2)(k+3)},$$

with

$$\mathcal{H}(k)=-\alpha(2k+3)+2(k+2)(k+3),$$

$$G(k)=2-\alpha-(k+1)(k+4),\ \ I(k)=-\alpha+(k+3)^2.$$

Therefore using the formula in Theorem B of Appendix 4, we have

$$A=\sum_{n=1}^{\infty}(-1)^n 2nt^{2n-1}a_{2n}$$

$$=\sum_{n=1}^{\infty}(-1)^n 2nt^{2n-1}\frac{1}{\delta^{2n}(2n)!}\sum_{k=2n}^{\infty}\frac{k!A_k\delta^k}{(k-2n)!}$$

$$=\sum_{k=2}^{\infty}\left[\sum_{n=1}^{[k/2]}\frac{k!(-1)^n}{(2n-1)!(k-2n)!}\left(\frac{t}{\delta}\right)^{2n}\frac{1}{t}\right]$$

$$\times A_k\delta^k=\sum_{k=-1}^{\infty}f_k(t)A_{k+3}=\sum_{k=-1}^{\infty}f_k(t)$$

$$\times\left[\frac{\Psi_k}{\zeta_0^{2k+3}}L(k)+\frac{\Psi_{k+2}}{\zeta_0^{2k+5}}M(k)+\frac{\Psi_{k+4}}{\zeta_0^{2k+7}}N(k)\right].$$
$$(8.19)$$

In the above equation $\Psi_k=0$, if $k\leq 0$ and

$$f_k(t)=\delta^{k+3}\sum_{n=1}^{[(k+3)/2]}\frac{(k+3)!(-1)^n}{(2n-1)!(k+3-2n)!}$$

$$\times\left(\frac{t}{\delta}\right)^{2n}\frac{1}{t}.$$

If we substitute the values of Ψ_k into (8.19),it reduces to

$$A=\int_0^1 g(u)\Omega_1(t,u)du,$$

where

$$\Omega_1(t,u)=\sum_{k=-1}^{\infty}f_k(t)\left[\frac{h_k(u)}{\zeta_0^{2k+3}}L(k)\right.$$

$$\left.+\frac{h_{k+2}(u)}{\zeta_0^{2k+5}}M(k)+\frac{h_{k+4}(u)}{\zeta_0^{2k+7}}N(k)\right].$$

In the above equation $h_k(u) = 0$, if $k \leq 0$ and

$$h_k(u) = \delta^k \sum_{n=0}^{[(k-1)/2]} \frac{(-1)^n k!}{(2n+1)!(k-2n-1)!} \times \left(\frac{u}{\delta}\right)^{2n+1}.$$

Also, using Theorem B of Appendix 4, we get

$$\sum_{n=0}^{\infty} (-1)^n b_{2n+1} t^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!\delta^{2n+1}}$$

$$\times \sum_{k=2n+1}^{\infty} \frac{k! B_k \delta^k}{(k-2n-1)!}$$

$$= \sum_{k=1}^{\infty} \left[\sum_{n=0}^{\left[\frac{k-1}{2}\right]} \frac{(-1)^n k!}{(2n+1)!(k-2n-1)!} \left(\frac{t}{\delta}\right)^{2n+1} \right] B_k \delta^k$$

$$= \sum_{k=-1}^{\infty} h_{k+2}(t) B_{k+2}$$

$$= \sum_{k=-1}^{\infty} h_{k+2}(t) \frac{2k+5}{\zeta_0^{k+1} \mathcal{H}(k)} \left[\frac{\Psi_k}{\zeta_0^{k+2}} (k+1)(k+3) \right.$$

$$\left. + \frac{\Psi_{k+2}}{\zeta_0^{k+4}} \frac{(k+3)(2k+3)}{2k+5} I(k) \right]$$

$$= \int_0^1 g(u) \Omega_2(t,u) du,$$

[0] where

$$\Omega_2(t,u) = \sum_{k=-1}^{\infty} h_{k+2}(t) \frac{2k+5}{\zeta_0^{k+1} \mathcal{H}(k)} \left[\frac{h_k(u)}{\zeta_0^{k+2}} (k+1) \right.$$

$$\times (k+3) + \frac{h_{k+2}(u)}{\zeta_0^{k+4}} \frac{(k+3)(2k+3)}{2k+5} I(k) \right].$$

In the above equation $h_k(u) = 0$, if $k \leq 0$. Thus (7.13) reduces to the following Fredholm integral equation of the second kind

$$g(t) + \int_0^1 g(u) K(t,u) du = h(t),$$

where

$$K(t,u) = -\frac{2}{\pi} \{\Omega_1(t,u) + \alpha \Omega_2(t,u)\}.$$

Appendix 1. Proof of (3.3). Since

$$\rho^2 = (\zeta \cos \vartheta - \delta)^2 + (\zeta \sin \vartheta)^2$$

$$= \zeta^2 \left\{ 1 - 2\cos\vartheta \frac{\delta}{\zeta} + \left(\frac{\delta}{\zeta}\right)^2 \right\},$$

$$\frac{1}{\rho} = \frac{1}{\zeta \sqrt{1 - 2\cos\vartheta \frac{\delta}{\zeta} + (\frac{\delta}{\zeta})^2}} = \sum_{n=0}^{\infty} P_n(\cos \vartheta) \frac{\delta^n}{\zeta^{n+1}},$$

$$\frac{P_n(\cos \theta)}{\rho^{n+1}} = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{1}{\rho}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial z^n} \left(\frac{P_k(\cos \vartheta)}{\zeta^{k+1}}\right) \delta^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^{n+k}}{\partial z^{n+k}} \left(\frac{1}{\zeta}\right) \frac{(-1)^k}{k!} \delta^k$$

$$= \sum_{k=0}^{\infty} \binom{n+k}{k} \frac{P_{n+k}(\cos \vartheta)}{\zeta^{n+k+1}} \delta^k.$$

Appendix 2. Proof of (3.16).

$$2\pi \rho^n P_n(\cos \theta) = \int_{-\pi}^{\pi} (z + ix \cos u + iy \sin u)^n du$$

$$= \int_{-\pi}^{\pi} (-\delta + Z + ix \cos u + iy \sin u)^n du$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} (-\delta)^{n-k}$$

$$\times \int_{-\pi}^{\pi} (Z + ix \cos u + iy \sin u)^k du$$

$$= 2\pi \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} (-\delta)^{n-k} \zeta^k P_k(\vartheta).$$

Appendix 3. Theorem A. If the equation

$$B_k = \sum_{i=0}^{k} \frac{k!}{(k-i)!i!} a_i,$$

is solved for $a_i' s$, it will be written as

$$a_i = \sum_{k=0}^{i} \frac{i!}{(i-k)!k!} B_k (-1)^{i-k}.$$

Proof. We prove it by mathematical induction. Suppose it is true for i, we will show that it is also true for $i+1$. Suppose B_{i+1} is given by

$$B_{i+1} = \sum_{k=0}^{i+1} \frac{(i+1)!}{(i+1-k)!k!} a_k = a_{i+1}$$

$$+ \sum_{k=0}^{i} \frac{(i+1)!}{(i+1-k)!k!} a_k.$$

Thus

$$a_{i+1} = B_{i+1} - \sum_{k=0}^{i} \frac{(i+1)!}{(i+1-k)!k!}$$

$$\times \sum_{j=0}^{k} B_j \frac{k!}{j!(k-j)!} (-1)^{k-j}$$

$$= B_{i+1} - \sum_{j=0}^{i} B_j \frac{(i+1)!}{j!} \sum_{k=j}^{i} \frac{(-1)^{k-j}}{(i+1-k)!(k-j)!}.$$
$$(A.1)$$

The inner summation in the above equation can written as, by changing the variable $k - j = m$

$$\sum_{k=j}^{i} \frac{(-1)^{k-j}}{(i+1-k)!(k-j)!}$$

$$= \sum_{m=0}^{i-j} \frac{(-1)^m}{(i+1-m-j)!m!}$$

$$= -\frac{(-1)^{i-j+1}}{(i-j+1)!},$$

^{0} since

$$\sum_{m=0}^{i-j+1} \frac{(-1)^m (i-j+1)!}{(i+1-m-j)!m!} = (1-1)^{i-j+1} = 0.$$

Then (A.1) is equal to

$$a_{i+1} = B_{i+1} + \sum_{j=0}^{i} \frac{B_i(i+1)!(-1)^{i+1-j}}{j!(i+1-j)!}$$

$$= \sum_{j=0}^{i+1} \frac{B_i(i+1)!(-1)^{i+1-j}}{j!(i+1-j)!}.$$

$\square$

Appendix 4. Theorem B. If the equation

$$B_k = \sum_{n=k}^{\infty} \binom{n}{k} (-\delta)^{n-k} a_n,$$

is solved for $a_n's$, it will be written as

$$a_n = \sum_{k=n}^{\infty} \binom{k}{n} \delta^{k-n} B_k.$$

Proof. Let

$$\sum_{n=0}^{\infty} a_n(-\delta)^n x^n = f(x),$$

then

$$a_n(-\delta)^n n! = f^{(n)}(0),$$

and

$$f^{(k)}(1) = \sum_{n=k}^{\infty} a_n(-\delta)^n \frac{n!}{(n-k)!} = k! B_k(-\delta)^k.$$

From Taylor's series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (x-1)^k.$$

Thus

$$a_n(-\delta)^n n! = f^{(n)}(0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} \frac{d^n}{dx^n} (x-1)^k \bigg|_{x=0}$$

$$= \sum_{k=n}^{\infty} \frac{f^{(k)}(1)}{(k-n)!} (-1)^{k-n} = \sum_{k=n}^{\infty} \frac{B_k(-1)^{k-n}(-\delta)^k k!}{(k-n)!}.$$

Therefore

$$a_n = \sum_{k=n}^{\infty} \frac{k!}{n!(k-n)!} B_k \delta^{k-n}.$$

$\square$

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