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Three-Term Linear Fractional Nabla Difference Equation

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ABSTRACT

In this present paper, a study on nabla difference equation and its third order linear fractional difference equation. A new generalized nabla difference equation is investigated from Three-term linear fractional nabla difference equation. A relevant example is proved and justify the proposed notions.

Keywords: *Fractional difference operator, nabla difference equation, linear fractional, Third-term equation.*

1. Introduction

In this present paper, we shall use the transform method to obtain solutions of a linear fractional nabla difference equation of the form

(1) $$\nabla_0^{\upsilon} x(t) + C_1 \nabla x(t) + C_2 x(t) = g(t),$$
$t = 1, 2, 3..,$

Where $1 < \upsilon \leq 2$. The fractional difference operator, ∇_0^{υ} is of R-L type and the operator ∇_0^{υ} is a Riemann-Liouville fractional difference operator, is defined by,

If $\mu > 0$, define the μ^{th}-term of fractional sum by

(2) $$\nabla_a^{-\mu} x(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma\mu} x(s)$$

Where $\rho(s) = s - 1$.

The aim for this paper is to develop and preserve the theory of linear fractional nabla difference equations as a corresponds of the theory of linear difference equations. We shall consider the three term equations, (1) is limited. An equation of the form

(3) $$\nabla_0^{2\mu} x(t) + C_1 \nabla_0^{\mu} x(t) + C_2 x(t) = g(t),$$
$t = 1, 2, 3..,$

is called a sequential fractional difference equation.

In general equation is,

(4) $$\nabla_0^{\upsilon_2} x(t) + C_1 \nabla_0^{\upsilon_1} x(t) + C_2 x(t) = g(t),$$
$t = 1, 2, 3..,$

Assume that $0 < \upsilon_1 \leq 1 < \upsilon_2 \leq 2$ as the only connection between υ_1 and υ_2. The operator of nabla is usually represents the backward difference operator and in this paper

(5) $$\nabla x(t) = x(t) - x(t-1),$$

$$\nabla^k x(t) = \nabla \nabla^{k-1} x(t), \quad k = 1, 2, 3..$$

The raising factorial power function is defined below,

(6) $$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(\alpha)}.$$

Then if $0 \leq m - 1 < \upsilon \leq m$, define by the Riemann-Liouville fractional difference equation is

(7) $$\nabla_c^{\upsilon} x(t) = \nabla^m \nabla_c^{\upsilon - m} x(t)$$

Where ∇^m denotes the standard m^{th} order nabla (backward) difference.

In section 2, we shall use the transform method to (1) and we find out the solutions. And the same time we shall expressed as a sufficient condition as a function of C_1 and C_2 for convergent of the solutions.

In section 3, we apply the algorithm in the case of a solution is 2^t and verified independently that the series represents the known function.

For further studying in this previous area, we refer the reader to the article on two-term linear fractional nabla difference equation [7].

2. Three-term Linear fractional nabla difference equation

In this section, we describe an algorithm to form a solution of an initial value problem for a three-term linear fractional nabla difference equation of the form,

(8) $\nabla_0^{\upsilon} x(t) + C_1 \nabla x(t) + C_2 x(t) = 0, \quad x(0) = c_0,$
$x(1) = c_1 \quad$ for $\quad t = 1, 2, 3 \ldots$

Where $1 < \upsilon \le 2$.

Apply the operator N_2 to the equation (8) we get,

$$N_2(\nabla_0^{\upsilon} x(t) + C_1 \nabla x(t) + C_2 x(t)) = 0$$

(9) $N_2(\nabla_0^{\upsilon} x(t)) + C_1 N_2(\nabla x(t)) + C_2 N_2 x(t) = 0$

First, consider a term $N_2(\nabla_0^{\upsilon} x(t))$ from (8) and use the result [7] we get,

"If $1 < \upsilon \le 2$,

$$N_{a+2}(\nabla_a^{\upsilon} f(t))(s) = s^{\upsilon} N_a(f(t))(s)$$
$$-s(1-s)^{a-1} f(a) - (1-s)^a \nabla_a^{\upsilon-1} f(a+1)$$
$$= s^{\upsilon} N_a(f(t))(s) - s(1-s)^{a-1} f(a)$$
$$-(1-s)^a (f(a+1) - (\upsilon-1) f(a))."$$

Which implies that,

$$N_2(\nabla_0^{\upsilon} x(t)) = s^{\upsilon} N_0(x(t)) - s(1-s)^{0-1} x(0)$$
$$-(1-s)^0 (x(1)) - (\upsilon-1) x(0))$$

(10) $N_2(\nabla_0^{\upsilon} x(t)) = s^{\upsilon} N_0(x(t))$
$$-s(1-s)^{-1} c_0 - (c_1 - (\upsilon-1) c_0)$$

Next, consider the term $N_2(\nabla x(t))$ on (9) and also, we know that the result [7],

"If $0 < \upsilon \le 1$, $\quad N_{a+1}\left(\nabla_a^{\upsilon} f(t)\right)(s) = s^{\upsilon} N_a\left(f(t)\right)(s)$
$$-(1-s)^{a-1} f(a)."$$

Which implies

$$N_2(\nabla x(t)) = N_2(\nabla x(t)) + \nabla x(1) - \nabla x(1)$$
$$= N_1(\nabla x(t)) - \nabla x(1)$$
$$= s N_1(x(t)) - (1-s)^{-1} c_0 - (c_1 - c_0)$$
$$= s N_0(x(t)) - \frac{1}{(1-s)} c_0 - c_1 + c_0 \frac{(1-s)}{(1-s)}$$
$$= s N_0(x(t)) - \frac{1}{(1-s)} c_0 - c_1 + \frac{1}{(1-s)} c_0 - \frac{s}{(1-s)} c_0$$

(11) $N_2(\nabla x(t)) = s N_0(x(t)) - \dfrac{s}{(1-s)} c_0 - c_1$

Similarly, we consider the last term,

$$N_2(x(t)) = N_2(x(t)) + c_1 - c_1 + \frac{1}{(1-s)} c_0 - \frac{1}{(1-s)} c_0$$
$$= N_2(x(t)) + c_1 + (1-s)^{-1} c_0 - c_1 + (1-s)^{-1} c_0$$

In particular

(12) $N_2(x(t)) = N_0(x(t)) - c_1 - (1-s)^{-1} c_0$

Substitute (10), (11) and (12) in (9) we get

$$N_2(\nabla_0^{\upsilon} x(t)) + C_1 N_2(\nabla x(t)) + C_2 N_2 x(t) = 0$$
$$s^{\upsilon} N_0(x(t)) - s(1-s)^{-1} c_0 - (c_1 - (\upsilon-1) c_0)$$
$$+C_1 \left(s N_0(x(t)) - \frac{s}{(1-s)} c_0 - c_1 \right)$$
$$+C_2 \left(N_0(x(t)) - c_1 - (1-s)^{-1} c_0 \right) = 0$$

$$(s^{\upsilon} + C_1 s + C_2) N_0(x(t)) - \frac{1}{(1-s)} (s + C_1 s + C_2) c_0$$
$$-(1 + C_1 + C_2) c_1 - (1-\upsilon) c_0 = 0$$

(13) $$N_0(x(t)) = \frac{(s + C_1 s + C_2)c_0}{(1-s)(s^{\upsilon} + C_1 s + C_2)} + \frac{(1 + C_1 + C_2)c_1 + (1-\upsilon)c_0}{(s^{\upsilon} + C_1 s + C_2)}$$

Now take $$\frac{1}{(s^{\upsilon} + C_1 s + C_2)} = \frac{1}{s^{\upsilon}\left(1 + \frac{C_1 s}{s^{\upsilon}} + \frac{C_2}{s^{\upsilon}}\right)}$$

$$= \frac{1}{s^{\upsilon}\left(1 + C_1 s^{1-\upsilon} + \frac{C_2}{s^{\upsilon}}\right)}$$

$$\frac{1}{(s^{\upsilon} + C_1 s + C_2)} = \frac{1}{s^{\upsilon}\left(1 + C_1 s^{1-\upsilon}\right)\left(1 + \frac{C_2}{s^{\upsilon}\left(1 + C_1 s^{1-\upsilon}\right)}\right)}$$ In general,

$$\frac{1}{s^{\upsilon} + C_1 s + C_2} = \sum_{n=0}^{\infty}(-1)^n s^{-\upsilon(n+1)} C_2^n \left(\frac{1}{1 + C_1 s^{1-\upsilon}}\right)^{n+1}$$

Note that,

$$\left(\frac{1}{1 + C_1 s^{1-\upsilon}}\right)^{n+1} = \sum_{m=n}^{\infty}(-1)^{m+n}\binom{m}{n}\left(C_1 s^{1-\upsilon}\right)^{m-n}$$

We get

$$\frac{1}{s^{\upsilon} + C_1 s + C_2} = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}(-1)^m \binom{m}{n} C_1^{m-n} s^{(1-\upsilon)m-n} C_2^n s^{-\upsilon(n+1)}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m \binom{m}{n} C_1^{m-n} C_2^n s^{m-m\upsilon-n+n\upsilon-n\upsilon-\upsilon}$$

(14) $$\frac{1}{s^{\upsilon} + C_1 s + C_2} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m \binom{m}{n} C_1^{m-n} C_2^n s^{((1-\upsilon)m)-n-\upsilon}$$

Since $$s^{((1-\upsilon)m)-n-\upsilon} = N_1\left(\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}\right)$$

$$= N_0\left(\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}\right).$$

By using the result [7], "
$AN_1 f(t) = -\frac{Af(0)}{1-s} + AN_0 f(t)$**".**

Now, the above equation is re-express N_1 as N_0, and we have,

$$\frac{1}{s^{\upsilon} + C_1 s + C_2} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m \binom{m}{n} C_1^{m-n} C_2^n \left(N_1\left(\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}\right)\right)$$

(15) $$= N_0 \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n} C_2^n \binom{m}{n} \left(\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}\right)$$

It follows from the result [7]
"$N_a f(t+1) = (1-s)^{-1} N_{a+1} f(t)$**".**

Which implies that, equation (15) we get,

$$\frac{(1-s)^{-1}}{s^{\upsilon} + C_1 s + C_2} = N_0 \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n} C_2^n \binom{m}{n}$$

(16) $$\left(\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}\right)$$

Moreover, from (14)

$$\frac{1}{s^{\upsilon} + C_1 s + C_2} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m \binom{m}{n} C_1^{m-n} C_2^n s^{(1-\upsilon)m-n-\upsilon} \cdot \frac{s}{s}$$

$$\frac{s}{s^{\upsilon} + C_1 s + C_2} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m \binom{m}{n} C_1^{m-n} C_2^n s^{(1-\upsilon)m-n-\upsilon+1}.$$

Similarly, we get

$$\frac{s}{s^{\upsilon}+C_1 s+C_2}=\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}N_0$$

$$\left(\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-2)}}}{\Gamma((\upsilon-1)m+n+(\upsilon-1))}\right)$$

and so

$$\frac{s(1-s)^{-1}}{s^{\upsilon}+C_1 s+C_2}=N_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}$$

(17) $$\left(\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-2)}}}{\Gamma((\upsilon-1)m+n+(\upsilon-1))}\right)$$

We consider an equation (13) we get

$$N_0 x(t)=\frac{(s+C_1 s+C_2)c_0}{(1-s)(s^{\upsilon}+C_1 s+C_2)}$$

$$+\frac{(1+C_1+C_2)c_1+(1-\upsilon)c_0}{(s^{\upsilon}+C_1 s+C_2)}$$

(18) $$N_0 x(t)=\frac{C_2 c_0}{(1-s)(s^{\upsilon}+C_1 s+C_2)}$$

$$+\frac{s(1+C_1)c_0}{(1-s)(s^{\upsilon}+C_1 s+C_2)}$$

$$+\frac{(1+C_1+C_2)c_1+(1-\upsilon)c_0}{(s^{\upsilon}+C_1 s+C_2)}$$

Substitute (15), (16) and (17) directly into (18) we get

$$N_0 x(t)=C_2 c_0 N_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$$

$$+(1+C_1)c_0 N_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-2)}}}{\Gamma((\upsilon-1)m+n+(\upsilon-1))}$$

$$+KN_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$$

$$x(t)=C_2 c_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$$

$$+(1+C_1)c_0\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-2)}}}{\Gamma((\upsilon-1)m+n+(\upsilon-1))}$$

$$+K\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$$

(19)

Where $K=((1+C_1+C_2)c_1+(1-\upsilon)c_0)$.

To simplify this representation, note that

$$\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}=\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}+\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-2)}}}{\Gamma((\upsilon-1)m+n+\upsilon-1)}$$

Since

$$\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}=\frac{\Gamma(t+1+(\upsilon-1)m+n+(\upsilon-1))}{\Gamma(t+1)\Gamma((\upsilon-1)m+n+\upsilon)}$$

$$=t\frac{\Gamma(t+(\upsilon-1)m+n+(\upsilon-1))}{\Gamma(t+1)\Gamma((\upsilon-1)m+n+\upsilon)}$$

$$+((\upsilon-1)m+n+(\upsilon-1))\frac{\Gamma(t+(\upsilon-1)m+n+(\upsilon-1))}{\Gamma(t+1)\Gamma((\upsilon-1)m+n+\upsilon)}$$

Thus, (19) can be expressed as

$$x(t)=K_1\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-2)}}}{\Gamma((\upsilon-1)m+n+(\upsilon-1))}$$

$$+K_2\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$$

(20)

Where

(21) $K_1=c_0(1+C_1+C_2)$,

$K_2=K+c_0C_2=(1+C_1+C_2)c_1+(1-\upsilon+C_2)c_0$ Note that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n}C_2^{n}\binom{m}{n}\frac{(t+1)^{\overline{(\upsilon-1)m+n+(\upsilon-2)}}}{\Gamma((\upsilon-1)m+n+(\upsilon-1))}\quad(22)$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}(-1)^m C_1^{m-n} C_2^n \binom{m}{n} \frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$$

(23)

are two linear independent solutions of (8). We give the details to obtain conditions for absolute convergence in (23) for fixed t.

First note that for each $t \geq 1$, $\frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$ is an increasing function in n. Now consider the ratio

$$\frac{t^{\overline{(\upsilon-1)m+n+1+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+1+\upsilon)} \Big/ \frac{t^{\overline{(\upsilon-1)m+n+(\upsilon-1)}}}{\Gamma((\upsilon-1)m+n+\upsilon)}$$

Then

$$\frac{\Gamma(t+(\upsilon-1)m+n+1+(\upsilon-1))}{\Gamma(t)\Gamma((\upsilon-1)m+n+1+\upsilon)} \frac{\Gamma(t)\Gamma((\upsilon-1)m+n+\upsilon)}{\Gamma(t+(\upsilon-1)m+n+(\upsilon-1))}$$

$$= \frac{t+(\upsilon-1)m+n+(\upsilon-1)}{(\upsilon-1)m+n+\upsilon} \geq 1$$

and the inequality is strict if $t > 1$.

Thus, compare (23) to

$$\sum_{m=0}^{\infty} \frac{t^{\overline{(\upsilon m+(\upsilon-1))}}}{\Gamma(\upsilon m+\upsilon)} \left|\sum_{n=0}^{m} C_1^{m-n} C_2^n \binom{m}{n}\right| = \sum_{m=0}^{\infty} \frac{(t)^{\overline{(\upsilon m+(\upsilon-1))}}}{\Gamma(\upsilon m+\upsilon)} |C_1+C_2|^m$$

and apply the ratio test. Thus, each of (22) and (23) are absolutely convergent if $|C_1+C_2|<1$.

Description of known functions:

We represent this method with an initial value problem for a classical second order finite difference equation. The unique solution of the initial value problem is $x(t)=2^t$, thus we obtain a series representation of 2^t as a linear combination of the forms (22) and (23).

Consider the initial value problem

(24) $10\nabla^2 x(t) - \nabla x(t) - 2x(t) = 0$, $t = 2,3,..$, $x(0)=1$, $x(1)=2$.

Then, apply (13) with

$$C_1 = \frac{-1}{10}, \quad C_2 = \frac{-1}{5}, \quad c_0 = 1, \quad c_1 = 2, \quad \upsilon = 2,$$

To obtain

$$N_0 x(t) = \frac{(s+(-1/10)s+(-1/5))1}{(1-s)(s^2+(-1/10)s+(-1/5))} + \frac{(1+(-1/10)+(-1/5))2+(1-2)1}{(s^2+(-1/10)s+(-1/5))}$$

$$N_0 x(t) = \frac{s-\frac{1}{10}s-\frac{1}{5}}{(1-s)\left(s^2-\frac{1}{10}s-\frac{1}{5}\right)} + \frac{2-\frac{2}{10}-\frac{2}{5}-1}{\left(s^2-\frac{1}{10}s-\frac{1}{5}\right)}$$

$$N_0 x(t) = \frac{\frac{9}{10}s-\frac{1}{5}}{(1-s)\left(s^2-\frac{1}{10}s-\frac{1}{5}\right)} + \frac{\frac{4}{10}}{\left(s^2-\frac{1}{10}s-\frac{1}{5}\right)}$$

From (21), $K_1 = 1\left(1-\frac{1}{10}-\frac{1}{5}\right) = \frac{7}{10}$

$$K_2 = \left(1-\frac{1}{10}-\frac{1}{5}\right)2 + \left(1-2-\frac{1}{5}\right)1 \quad = \left(\frac{7}{10}\right)2 - \frac{6}{5} = \frac{1}{5}.$$

and

$$x(t) = \frac{7}{10}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n \left(\frac{1}{10}\right)^{m-n} \binom{m}{n} \frac{(t+1)^{\overline{m+n}}}{\Gamma(m+n+1)} + \frac{1}{5}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n \left(\frac{1}{10}\right)^{m-n} \binom{m}{n} \frac{(t)^{\overline{m+n}}}{\Gamma(m+n+2)}.$$

(25)

The unique solution of (24) is 2^t and the series given in (25) absolutely convergent for all $t = 0,1,2,..$ Write (25) as $x(t) = \frac{7}{10}C(t) + \frac{1}{5}D(t)$.

Where $C(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n \left(\frac{1}{10}\right)^{m-n} \binom{m}{n} \frac{(t+1)^{\overline{m+n}}}{\Gamma(m+n+1)}$

$D(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n \left(\frac{1}{10}\right)^{m-n} \binom{m}{n} \frac{(t)^{\overline{m+n}}}{\Gamma(m+n+2)}$ To prove $x(t+1) = 2x(t)$, or

$$\frac{7}{10}C(t+1)+\frac{1}{5}D(t+1)=2\left(\frac{7}{10}C(t)+\frac{1}{5}D(t)\right)$$

It is sufficient to show that

$D(t+1)=C(t)+D(t)$ and

$$\frac{7}{10}C(t+1)=\left(\frac{6}{5}C(t)+\frac{1}{5}D(t)\right).$$

Now consider $D(t+1)$,

$$D(t+1)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}\frac{(t+1)^{\overline{m+n+1}}}{\Gamma(m+n+2)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}\frac{\Gamma(t+1+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}(t+m+n+1)\frac{\Gamma(t+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}t\frac{\Gamma(t+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}(m+n+1)\frac{\Gamma(t+m+n+1)}{\Gamma(t+1)\Gamma(m+n+2)}$$

$$D(t+1)=D(t)+C(t).$$

We have yet to obtain a direct approach to take $C(t+1)$. We begin by showing directly that $D(t)$ satisfies

$10\nabla^2x(t)-\nabla x(t)-2x(t)=0,\qquad t=2,3,.$

Apply the power rule and

$$D(t)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}\frac{t^{\overline{m+n+1}}}{\Gamma(m+n+2)}$$

$$\nabla D(t)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}$$

$$\nabla^2D(t)=\sum_{m=1}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}\frac{t^{\overline{m+n-1}}}{\Gamma(m+n)}$$

Thus,

$$\nabla^2D(t)=\sum_{m=0}^{\infty}\sum_{n=0}^{m+1}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m+1-n}\binom{m+1}{n}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}$$

$$=\sum_{m=0}^{\infty}(\frac{1}{10}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\left(\binom{m}{n}+\binom{m}{n-1}\right)\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}+\left(\frac{1}{5}\right)^{m+1}\frac{t^{\overline{2m+1}}}{\Gamma(2m+2)})$$

$$=\sum_{m=0}^{\infty}\frac{1}{10}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{m+1}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n-1}\frac{t^{\overline{m+n}}}{\Gamma(m+n+1)}$$

$$=\frac{1}{10}\nabla D(t)+\frac{1}{5}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(\frac{1}{5}\right)^n\left(\frac{1}{10}\right)^{m-n}\binom{m}{n}\frac{t^{\overline{m+n+1}}}{\Gamma(m+n+2)}$$

$$\nabla^2D(t)=\frac{1}{10}\nabla D(t)+\frac{1}{5}D(t)$$

A similar calculation shows that $C(t)$ satisfies

$10\nabla^2x(t)-\nabla x(t)-2x(t)=0,\qquad t=2,3,.$

We close by arguing that

$$\frac{7}{10}C(t+1)=\left(\frac{6}{5}C(t)+\frac{1}{5}D(t)\right).$$

Simplify

$$10\nabla^2C(t+1)-\nabla C(t+1)-2C(t+1)=0$$

$$10(C(t+1)-2C(t)+C(t-1))$$

$$-(C(t+1)-C(t))-2C(t+1)=0$$

To obtain $\frac{7}{10}C(t+1)=\frac{19}{10}C(t)-C(t-1)$

$$=\frac{6}{5}C(t)+\left(\frac{7}{10}C(t)-C(t-1)\right)$$

Now it is sufficient to show that

$$\left(\frac{7}{10}C(t)-C(t-1)\right)=\frac{1}{5}D(t).$$

Employ $C(t) = D(t+1) - D(t)$ and $C(t-1) = D(t) - D(t-1)$ to obtain

$$\left(\frac{7}{10}C(t) - C(t-1)\right) = \frac{7}{10}(D(t+1)$$

$$-D(t)) - (D(t) - D(t-1))$$

$$= \frac{7}{10}(D(t+1) - \frac{17}{10}D(t) + D(t-1)$$

$$= \left(\frac{7}{10}(D(t+1) - \frac{19}{10}D(t) + D(t-1)\right) + \frac{2}{10}D(t)$$

$$\left(\frac{7}{10}C(t) - C(t-1)\right) = \frac{1}{5}D(t).$$

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